BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 79, Number 4, July 1973

ON A CONJECTURE OF M. KAC[1]

BY DANIEL W. STROOCK

Communicated by I. M. Singer, January 11, 1973

Introduction. Consider the region $\mathscr{G} = \{(x, y, z) \in R^3 : x^2 + y^2 \leqq f^2(z), 0 \leqq z \leqq L\}$ obtained by rotating the curve $f(z), 0 \leqq x \leqq L$, about the z-axis. Denote by $q(t, \vec{r}, \vec{r}')$ the fundamental solution of the heat equation in $\mathscr{G}$ with zero boundary value on $\partial\mathscr{G}$, and let $p(t, \vec{r}, \vec{r}') = (1/(2\pi t)^{3/2}) \cdot \exp(-|\vec{r} - \vec{r}'|^2/2t)$. Express everything in terms of cylindrical coordinates (ρ, θ, z). The purpose of this note is to show that, for each $m \geqq 0$,

$$\frac{\displaystyle\int_0^L dz \int_0^{f(z)} \rho\, d\rho \left.\frac{\partial^{2m}}{\partial\theta^{2m}} q(t, (\rho, 0, z), (\rho, \theta, z))\right|_{\theta=0}}{\displaystyle\int_0^L dz \int_0^{f(z)} \rho\, d\rho \left.\frac{\partial^{2m}}{\partial\theta^{2m}} p(t, (\rho, 0, z), (\rho, \theta, z))\right|_{\theta=0}} \to 1 \tag{1}$$

as $t \searrow 0$. As Kac points out in [**2**], this equation enables one to recapture the distribution of the function $f(z)$ from the eigenvalues of the Laplacian, with zero boundary conditions, in $\mathscr{G}$. Hence, one gets a refined version of Weyl's theorem (cf. Kac [**1**]) by taking advantage of the cylindrical symmetry of $\mathscr{G}$. Indeed, Weyl's theorem results from (1) with $m = 0$. As will be seen, (1) can be viewed as an extended "principle of not feeling the boundary."

The "principle of not feeling the boundary" is the statement that $q(t, \vec{r}, \vec{r})/p(t, \vec{r}, \vec{r}) \to 1$ as $t \searrow 0$ for $\vec{r} \in \mathscr{G}$. Combining this with the maximum principle, which tells us that $q(t, \vec{r}, \vec{r}) \leqq p(t, \vec{r}, \vec{r})$, one can easily derive

$$\int_{\mathscr{G}} q(t, \vec{r}, \vec{r})\, d\vec{r} \Big/ \int_{\mathscr{G}} p(t, \vec{r}, \vec{r})\, d\vec{r} \to 1.$$

The derivation of (1) can be accomplished in the same way, only there is now an extra complication. The maximum principle can no longer be invoked to compare the θ-derivatives of q and p. For this reason we introduce a special representation of q and p. From this representation it will be possible to conclude that

AMS (*MOS*) *subject classifications* (1970). Primary 35P15, 35J25; Secondary 60J65.
[1]Results obtained at the Courant Institute of Mathematical Sciences, New York University, with the Air Force Office of Scientific Research Grant AF-AFOSR-71-2055.