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APPROXIMATION AND WEAK-STAR APPROXIMATION IN BANACH SPACES[1]

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ABSTRACT. If X^* has a weak-star basis and if X is separable, then X has a basis. If X^* has the weak-star λ-m.a.p. [a weak-star π_λ-decomposition], then X has the λ-m.a.p. [a $\pi_{\lambda+\lambda^2+\varepsilon}$-decomposition]. If X^* has a weak-star π_λ-decomposition and if X is separable, then X has a finite dimensional decomposition.

The problem of whether X is separable if X^* has a weak-star basis [**8**, p. 151] is unsolved, though there are candidates for a counterexample [3, 3.1], [**6**, pp. 243, 244]. In this note techniques developed in [**5**] are used, together with certain properties of weak convergence, to show that weak-star approximation methods in X^* will yield approximation properties in X.

In [**5**] the authors established very deep relationships between approximation methods in a Banach space X and its dual X^*. In particular they proved that if X^* has a basis, then X has a shrinking basis; and if X^* is a π_λ-space, then X is a π_δ-space for some $\delta > 1$. A fundamental tool in this work was the "principle of local reflexivity" [**5**], [**6**]. The basic corollary needed below is the following Theorem A [**5**, 3.1] or [**4**, p. 482], where $\mathscr{L}(B)$ is the space of bounded linear operators from B to B.

THEOREM A. *Let T be a finite rank operator in $\mathscr{L}(X^*)$ and let $F \subset X^*$ have* $\dim F < \infty$. *Let $\varepsilon > 0$. Then there is an S in $\mathscr{L}(X)$ such that $S^*(X^*) = T(X^*)$, $f(Sx) = Tf(x)$ for each f in F, x in X, and $\|S\| \leqq (1 + \varepsilon)\|T\|$. If T is a projection, then taking F to include $T(X^*)$, S is a projection.*

THEOREM 1. *Let (T_α) be a net of finite rank operators in $\mathscr{L}(X^*)$ such that $\|T_\alpha\| \leqq \lambda$ for all α and* $\lim T_\alpha f(x) = f(x)$ *for each f in X^*, x in X. Then there is a net of finite rank operators (S_β) in $\mathscr{L}(X)$ such that* $\lim S_\beta x = x$ *for each x, $\|S_\beta\| \leqq \lambda$ for each β.*

PROOF. For each finite-dimensional subspace F of X^*, use Theorem A to find $S_{\alpha,F}$ such that $f(S_{\alpha,F}x) = T_\alpha f(x)$ for every f in F, x in X, $S^*_{\alpha,F}(X^*) = T_\alpha(X^*)$ and $\|S_{\alpha,F}\| \leqq \lambda(1 + 1/(1 + \dim F))$. Let $(\alpha_1, F_1) \geqq (\alpha_2, F_2)$, if $\alpha_1 \geqq \alpha_2$, $F_1 \supset F_2$. Then $(1 + \dim(F))S_{\alpha,F}/(2 + \dim(F)) = R_{\alpha,F}$ has norm $\leqq \lambda$ and $\lim f(R_{\alpha,F}x) = f(x)$ for every f in X^*, x in X. Then a net(P_β) of convex combinations of $(R_{\alpha,F})$ has the property that $\lim P_\beta x = x$ for

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