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GROTHENDIECK AND WHITEHEAD GROUPS OF TORSION FREE ABELIAN GROUPS

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Let $\mathscr{A}$ denote the category of torsion free abelian groups of finite rank and let $K_0(\mathscr{A})$ and $K^0(\mathscr{A})$ be the Grothendieck groups of $\mathscr{A}$ (modulo split exact sequences and exact sequences, respectively).

J. Rotman [5] determined the group and ring structure of $K^0(\mathscr{A})$; in particular, $K^0(\mathscr{A})$ is a free abelian group of uncountable rank. There is a canonical epimorphism $\pi_0 : K_0(\mathscr{A}) \to K^0(\mathscr{A})$ so $K_0(\mathscr{A})$ has a free summand.

Let $\mathscr{C}$ be the full subcategory of $\mathscr{A}$ consisting of groups with constant p-rank (i.e., there is an integer n such that the Z/pZ-dimension of A/pA is equal to n for all primes p). Define $K_0(\mathscr{C})$ to be the Grothendieck group of $\mathscr{C}$ (modulo split exact sequences) and let $\tilde{K}_0(\mathscr{C})$ be the kernel of the rank homomorphism from $K_0(\mathscr{C})$ to Z, the ring of integers.

PROPOSITION 1. *$\tilde{K}_0(\mathscr{C})$ is isomorphic to the kernel of π_0.*

The category $\mathscr{C}'$ is defined by letting the objects of $\mathscr{C}'$ be the objects of $\mathscr{C}$, and with morphism sets $Q \otimes_Z \mathrm{Hom}_Z(A, B)$ for groups A and B in $\mathscr{C}'$. There is a canonical epimorphism $\sigma_0 : K_0(\mathscr{C}) \to K_0(\mathscr{C}')$. Moreover, $K_0(\mathscr{C}')$ is a free abelian group (of uncountable rank) since $\mathscr{C}'$ has a Krull-Schmidt theorem (e.g., see Walker [6]).

If R is a ring with identity, then $K_0(R)$ is defined to be $K_0(\mathscr{P}_R) = K^0(\mathscr{P}_R)$, $\mathscr{P}_R$ the category of finitely generated projective R-modules.

A corollary to the next theorem is: The torsion subgroup of $K_0(\mathscr{A})$ is nonzero (for $K_0(R) \simeq Z \oplus I(R)$, where $I(R)$ is the ideal class group of R).

THEOREM 2. *Suppose that R is a Dedekind domain such that R^+, the additive group of R, is a reduced torsion free abelian group of finite rank. Then $K_0(R)$ is isomorphic to a subgroup of $K_0(\mathscr{A})$.*

Let $K_1(\mathscr{A})$ be the Whitehead group of $\mathscr{A}$ (as defined by Bass [3]). Since $\mathscr{C}$ is a cofinal subcategory of $\mathscr{A}$, we have

COROLLARY 3. $K_1(\mathscr{C}) \simeq K_1(\mathscr{A})$.

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