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A COMPLETE BOOLEAN ALGEBRA OF SUBSPACES WHICH IS NOT REFLEXIVE

BY JOHN B. CONWAY

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This note provides a negative answer to a question raised by P. R. Halmos [**2**, Problem 9]. For the convenience of the reader, the terminology necessary to understand the question is presented here. Let $\mathscr{L}$ be a lattice of subspaces of a Hilbert space $\mathscr{H}$ and let $\operatorname{Alg} \mathscr{L}$ be the algebra of all bounded operators in $\mathscr{B}(\mathscr{H})$ that leave each subspace in $\mathscr{L}$ invariant. If $\mathscr{A} \subset \mathscr{B}(\mathscr{H})$, let $\operatorname{Lat} \mathscr{A}$ be the lattice of all subspaces of $\mathscr{H}$ that are left invariant by each operator in $\mathscr{A}$. A lattice $\mathscr{L}$ is *reflexive* if $\operatorname{Lat} \operatorname{Alg} \mathscr{L} = \mathscr{L}$. If $\mathscr{L}$ is a reflexive lattice and $\{P_i\}$ is a net of orthogonal projections such that $P_i(\mathscr{H}) \in \mathscr{L}$ for each i and $P_i \to P$ in the strong operator topology then $P(\mathscr{H}) \in \mathscr{L}$; in other words, $\mathscr{L}$ is *strongly closed*. It is true that a strongly closed lattice of subspaces is a complete lattice, but the converse is false.

A Boolean algebra of subspaces is a distributive lattice $\mathscr{L}$ such that for each M in $\mathscr{L}$ there is a unique M' in $\mathscr{L}$ such that $M \cap M' = (0)$ and $M \vee M' \equiv (M + M')^- = \mathscr{H}$. (Note that it is only required that $\mathscr{H}$ be the closure of $M + M'$.) Problem 9 of [**2**] asks: Is every complete Boolean algebra of subspaces reflexive? The answer is no, and this is shown in this paper by giving a complete Boolean algebra of subspaces which is not strongly closed. In one sense this answer seems unsatisfactory because a new question arises: Is every strongly closed Boolean algebra of subspaces reflexive? In another sense the answer is satisfying because the original question was the proper one to ask. The property of completeness is a lattice theoretic one, while the property of being strongly closed is not.

For the remaining terminology the reader is referred to [**4**] and other standard references. If $X = [0, 2\pi]$, let μ be a positive singular measure on the collection $\mathscr{A}$ of Borel subsets of X. For A in $\mathscr{A}$ define

$$\varphi_A(z) = \exp\left(-\int_A \frac{e^{i\theta} + z}{e^{i\theta} - z}\, d\mu(\theta)\right), \qquad |z| < 1,$$

and put $\varphi = \varphi_X$. Each φ_A is an inner function, and φ_A is a divisor of φ_B if and only if $A \subset B$. $\mathscr{H} = H^2 \ominus \varphi H^2$ and, for each A in $\mathscr{A}$, $M_A = \varphi_A H^2 \ominus \varphi H^2$.

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