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KRULL DIMENSION-NILPOTENCY AND GABRIEL DIMENSION

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One property of right noetherian rings and noetherian modules is that they have a Krull dimension in the sense of Rentschler-Gabriel [**11**]; see below for a definition. This property is considerably weaker than being noetherian. Nevertheless it was shown in [**3**] and [**8**] that, if a semiprime ring has a Krull dimension, then it is a right Goldie ring; and it has been conjectured that, in any ring with Krull dimension, nil subrings are nilpotent. Here we describe a proposition by means of which we have established the truth of this conjecture; and we describe other results which follow from it. Several of these are concerned with the connection between the Krull dimension of a module and its "Krull dimension" as defined in [**1**], which we call its Gabriel dimension. The detailed proofs will appear in [**4**], [**6**] and [**10**].

The proposition, which is basic to the proofs of the results which follow, deals with a module M which has a descending chain of submodules, of order type ω,

$$M_0 \supseteq M_1 \supseteq M_2 \supseteq \ldots$$

and an ascending chain of submodules, of order type ε for some ordinal ε,

$$B_1 \subseteq B_2 \subseteq \cdots \subseteq B_\delta \ldots .$$

Let us write $\delta M_n = B_\delta \cap M_n$. From the two chains above, we obtain the following array.

Note that, by the construction of the array, if $\delta \geqq \delta'$ then $\delta M_n \supseteq \delta' M_n + \delta M_{n+1}$ for each n.

PROPOSITION 1. *Suppose that, with the situation as described above, there is a countably infinite sequence of ordinals $\delta_0 < \delta_1 < \delta_2 < \cdots < \varepsilon$ such that, for each n, $\delta_{n+1} M_n \supsetneqq \delta_n M_n + \delta_{n+1} M_{n+1}$. Then the module*

$$\sum_{n=0}^{\infty} \delta_{n+1} M_n \Big/ \sum_{n=0}^{\infty} \delta_n M_n$$

is the infinite direct sum of its nonzero submodules

$$\delta_{n+1} M_n + \sum_{n=0}^{\infty} \delta_n M_n \Big/ \sum_{n=0}^{\infty} \delta_n M_n.$$

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