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SELFADJOINT SUBSPACE EXTENSIONS OF NONDENSELY DEFINED SYMMETRIC OPERATORS[1]

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1. **Subspaces in** $\mathfrak{H}^2$. Let $\mathfrak{H}$ be a Hilbert space over the complex field C, and let $\mathfrak{H}^2 = \mathfrak{H} \oplus \mathfrak{H}$ be the Hilbert space of all pairs $\{f, g\}$, where $f, g \in \mathfrak{H}$, with the inner product $(\{f, g\}, \{h, k\}) = (f, h) + (g, k)$. A *subspace* T in $\mathfrak{H}^2$ is a closed linear manifold in $\mathfrak{H}^2$; its *domain* $\mathfrak{D}(T)$ is the set of all $f \in \mathfrak{H}$ such that $\{f, g\} \in T$ for some $g \in \mathfrak{H}$, and its *range* $\mathfrak{R}(T)$ is the set of all $g \in \mathfrak{H}$ such that $\{f, g\} \in T$ for some $f \in \mathfrak{H}$. For $f \in \mathfrak{D}(T)$ we put $T(f) = \{g \in \mathfrak{H} | \{f, g\} \in T\}$. A subspace T in $\mathfrak{H}^2$ is the graph of a linear function if $T(0) = \{0\}$; in this case we say T is an *operator* in $\mathfrak{H}$, and then we denote $T(f)$ by Tf.

The *adjoint* T^* of a subspace T in $\mathfrak{H}^2$ is defined by

$$T^* = \{\{h, k\} \in \mathfrak{H}^2 | (g, h) = (f, k) \text{ for all } \{f, g\} \in T\}.$$

If J is the unitary operator in $\mathfrak{H}^2$ given by $J\{f, g\} = \{g, -f\}$, then $T^* = \mathfrak{H}^2 \ominus JT$, the orthogonal complement of JT in $\mathfrak{H}^2$. This shows that T^* is also a subspace in $\mathfrak{H}^2$.

If T is a subspace in $\mathfrak{H}^2$, let $T_\infty = \{\{f, g\} \in T | f = 0\}$. Then $T_s = T \ominus T_\infty$ is a closed operator in $\mathfrak{H}$, and we have the orthogonal decomposition $T = T_s \oplus T_\infty$, with $\mathfrak{D}(T_s)$ dense in $\mathfrak{H} \ominus T^*(0)$, $\mathfrak{R}(T_s) \subset \mathfrak{H} \ominus T(0)$.

A *symmetric subspace* S in $\mathfrak{H}^2$ is one satisfying $S \subset S^*$, and a *selfadjoint subspace* H is a symmetric one such that $H = H^*$. If $H = H_s \oplus H_\infty$ is a selfadjoint subspace in $\mathfrak{H}^2$ we have the result (due to Arens, [**1**, Theorem 5.3]) that H_s, considered as an operator in $\mathfrak{H} \ominus H(0)$, is a densely defined selfadjoint operator there. This permits a spectral analysis of a selfadjoint subspace H, once its operator part H_s and its purely multi-valued part H_∞ have been identified.

If S, S_1 are symmetric subspaces in $\mathfrak{H}^2$ such that $S \subset S_1$, then S_1 is said to be a *symmetric extension* of S. In [**3**] (see also [**2**]) we described all symmetric and selfadjoint extensions of a symmetric subspace S in $\mathfrak{H}^2$. In this note we characterize precisely, in terms of "generalized boundary conditions", those selfadjoint subspace extensions of a nondensely defined symmetric operator S in $\mathfrak{H}$. Applications to ordinary differential operators will be indicated in a subsequent note. Detailed

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