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STABLE RINGS AND A PROBLEM OF BASS

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In [1] Bass proved the following theorem. Let A be a 1-dimensional Noetherian ring with finite integral closure.[1] If every ideal of A can be generated by two elements, then every ideal is projective over its endomorphism ring. Bass also conjectured that the converse holds. Theorem 2.4 below shows that this is indeed the case.

1. **Stable ideals and stable rings.** Let A be a commutative Noetherian ring with 1. In analogy with Lipman's terminology in [3], we call an ideal I of a ring A *stable* if I is projective over its endomorphism ring, $\mathrm{End}_A(I)$. The ring A is *stable* if every ideal of A is stable. Since the relationship between stability and multiplicity is one key factor in the proof of the theorem, we gather some of the needed facts here.

Let I be an ideal of A which contains a nonzero divisor. Then $\mathrm{End}_A(I) \cong (I:I) = \{x \in K | xI \subseteq I\}$, where K is the total quotient ring of A. Thus $\mathrm{End}_A(I)$ is a subring of the integral closure of A in K and I is an ideal of $\mathrm{End}_A(I)$.

Suppose that A is a 1-dimensional local Macaulay ring and that I is an ideal containing a nonzero divisor. Then I is a stable ideal of A if and only if I is a principal ideal of the semilocal ring $\mathrm{End}_A(I)$. Therefore, if I is stable, $\mathrm{End}_A(I) = A^I$, the ring obtained from A by blowing up I (cf. [3]). For reference, we state as Proposition 1.1 the characterization of stable ideals due to Lipman [3]. The notation is as follows: $\mu(I)$ denotes the multiplicity of an ideal I and $\lambda(B)$ denotes the length of an A-module B.

PROPOSITION 1.1 (LIPMAN). *Let A be a 1-dimensional local Macaulay ring. The following conditions are equivalent for an ideal I containing a nonzero divisor.*

(1) *I is stable;*
(2) *$\lambda(I^n/I^{n+1}) = \mu(I)$, for all $n > 0$;*
(3) *$\lambda(A/I^n) = \mu(I)n - \lambda((I:I)/A)$, for all $n > 0$.*

2. **Proof of Bass' conjecture.** We will need the following proposition which turns out to be a special case of a theorem of Rees [5]. We give a simple direct proof.

PROPOSITION 2.1. *Let A be a 1-dimensional local Macaulay ring with*

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[1] Here, and in the statement of Theorem 2.4, A is assumed to have no nonzero nilpotent elements.