BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 79, Number 3, May 1973

GEOMETRY OF LEBESGUE-BOCHNER FUNCTION SPACES—SMOOTHNESS

BY I. E. LEONARD AND K. SUNDARESAN[1]

Communicated by Robert G. Bartle, October 27, 1972

ABSTRACT. There exist real Banach spaces E such that the norm in E is of class C^∞ away from zero; however, for any p, $1 \leqq p \leqq \infty$, the norm in the Lebesgue-Bochner function space $L_p(E,\mu)$ is not even twice differentiable away from zero. It is this fact that led to a deeper study of the order of differentiability of the norm function in the spaces $L_p(E, \mu)$, and the main objective of this paper is to announce the complete determination of the order of smoothness of the norm in this class of Banach spaces.

1. **Introduction.** The class of Lebesgue-Bochner function spaces is discussed in Bochner and Taylor [**1**] and has been found to be of considerable importance in various branches of analysis. For their importance in Fourier analysis, the reader is referred to [**1**], and Stein and Weiss [**11**]. Various geometric properties of these spaces have been discussed in Day [**5**], Köthe [**9**], and McShane [**10**]; however, there has been no discussion of higher order smoothness of the norm in this class of Banach spaces. The only known result concerning smoothness is due to McShane [**10**], and his result concerns only the directional derivative (Gâteaux derivative) of the norm. In this paper, a complete characterization of k-times (continuous) differentiability of the norm in $L_p(E, \mu)$ is provided. It might be noted here that the order of smoothness of the Banach spaces $L_p(\mu)$ has been discussed in Sundaresan [**12**], and related results can be found in Bonic and Frampton [**2**].

2. **Definitions and notation.** The definitions and notation used throughout the paper are collected in this section for easy reference. In the following, E denotes a real Banach space. The unit ball of E is $U = \{x \in E \mid \|x\| \leqq 1\}$, and its boundary $S = \{x \in E \mid \|x\| = 1\}$ is the unit sphere of E. In what follows, (T, Σ, μ) is an arbitrary measure space, where μ is an extended non-negative real-valued measure. To avoid trivialities it is assumed that the range of μ contains at least one nonzero real number, and μ is not supported by finitely many atoms.

2.1. DEFINITION. If $1 \leqq p < \infty$, let

AMS (MOS) subject classifications (1970). Primary 46E40, 28A45; Secondary 58C20, 28A15.

Key words and phrases. Lebesgue-Bochner function spaces, higher-order differentiability, smoothness.

[1] Research supported in part by a Scaife Faculty Grant administered by Carnegie-Mellon University.

Copyright © American Mathematical Society 1973