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DIMENSION SEQUENCES FOR COMMUTATIVE RINGS

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Let R be a commutative ring with identity of finite (Krull) dimension n_0, and for each positive integer k, let n_k be the dimension of the polynomial ring $R^{(k)} = R[X_1, \ldots, X_k]$ in k indeterminates over R. The sequence $\{n_i\}_{i=0}^{\infty}$ is called *the dimension sequence for* R, and the sequence $\{d_i\}_{i=1}^{\infty}$, where $d_i = n_i - n_{i-1}$ for each i, is called *the difference sequence for* R. We are concerned with a determination of those sequences of nonnegative integers that can be realized as the dimension sequence of a ring.

Several restrictions on the sequences $\{n_i\}$ and $\{d_i\}$ are known. For example, $n_k + 1 \leqq n_{k+1} \leqq 2n_k + 1$ for each positive integer k [**5**], and $n_0 + k \leqq n_k \leqq (n_0 + 1)(k + 1) - 1$ [**4**]. In particular, the only dimension sequence for a zero-dimensional ring is $0, 1, 2, \ldots$. Jaffard in [**4**] proved that the difference sequence for R is eventually a constant less than or equal to $n_0 + 1$—that is, there is a positive integer k such that $n_0 + 1 \geqq d_k = d_{k+1} = \cdots$. Moreover, if R is an integral domain, then both the integer k and the value d_k relate to the valuative dimension $\dim_v R$, defined to be $\sup\{\text{rank } V | V \text{ is a valuation overring of } R\}$. Jaffard proved that the eventual value of the difference sequence for R is 1 if and only if R has finite valuative dimension. In fact, if $\dim_v R = k < \infty$, then Jaffard proved that $d_i = 1$ for $i \geqq k + 1$; in [**1**], Arnold improved Jaffard's bound by 1 to $i \geqq k$, and this bound, in turn, cannot be improved in the general case.

We are able to prove that the restrictions mentioned in the previous paragraph are all that are necessary in order that an increasing sequence $\{m_i\}_{i=0}^{\infty}$ of positive integers with nonincreasing difference sequence $\{m_i - m_{i-1}\}_{i=1}^{\infty}$ should be the dimension sequence of an integral domain. It is known, however, that the difference sequence for an integral domain need not be nonincreasing, and hence we require additional restrictions to solve the general problem.

We denote by $\mathscr{S}$ the set of sequences $\{m_i\}_{i=0}^{\infty}$ of positive integers such that the corresponding difference sequence $\{t_i\}_{i=1}^{\infty}$, where $t_i = m_i - m_{i-1}$, satisfies the following conditions:

(1) $m_0 + 1 \geqq t_1 \geqq t_2 \geqq \cdots$.

(2) There is a positive integer k such that $1 \leqq t_k = t_{k+1} = t_{k+2} = \cdots$.

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