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FILTRATION TECHNIQUES IN THE STUDY OF LIE ALGEBRAS

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1. **Introduction.** In the present note we announce some results which have been obtained by the systematic exploitation of filtration arguments in the study of (infinite dimensional) Lie algebras. The fundamental theorem along this line appears in the following section. In §3 we indicate how this result can be used to provide a new proof and at the same time a generalization of the Shirshov–Witt subalgebra theorem; we recall that this theorem asserts that, with ground ring a field, a subalgebra of a free Lie algebra is free (this being of course the Lie-theoretic analogue of the Nielsen–Schreier subgroup theorem). §4 contains the statement of a result which will be seen to be the coproduct analogue of M. Hall's basis theorem for a free Lie algebra. From the existence of such a coproduct basis one is easily led to examples which show that there is no (at least strict) Lie-theoretic analogue of the Kurosch subgroup theorem; the problem of the structure of the subalgebras of a coproduct of Lie algebras takes on added interest. Some results along this line are listed in §5; this section contains also a criterion for the freeness of a Lie algebra. The sixth and final section contains acknowledgments and remarks.

2. **Setting of the context and statement of the fundamental theorem.** It is to be expected that as we employ filtration arguments we should eventually find ourselves dealing with graded objects. In order to unify the presentation we have elected to assume from the beginning that our objects are graded. We let J denote a monoid whose operation is written additively and whose set of nonzero elements is denoted by J^o, and R a commutative ring with unit. The notions of J-graded R-module, J-graded R-algebras, etc. are defined in the usual way. The sign commutation rule requires that the notion of odd and even be defined for the elements of J. We are led in this way to the notion of an oriented monoid: An oriented monoid J is a pair consisting of a commutative monoid, also denoted by J, with the cancellation property and a morphism $\varepsilon: J \to \boldsymbol{Z}/2\boldsymbol{Z}$ of monoids, where $\boldsymbol{Z}$ denotes the monoid of rational integers. The morphism ε is called the orientation of J; an element s in J is called even if $\varepsilon(s) = 0$ and odd if

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