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A FUNDAMENTAL SOLUTION FOR A SUBELLIPTIC OPERATOR[1]

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1. **Introduction.** Let $\mathscr{L}: C^\infty(M) \to C^\infty(M)$ be a formally selfadjoint differential operator of order 2 on the Riemannian manifold M. $\mathscr{L}$ is said to be *subelliptic of order* ε $(0 < \varepsilon < 1)$ at $x \in M$ if there exist a neighborhood V of x and a constant $c > 0$ such that for all $u \in C_0^\infty(V)$,

$$(1) \qquad \|u\|_\varepsilon^2 \leqq c(|(\mathscr{L}u, u)| + \|u\|^2),$$

where $\|u\|$ is the L^2 norm and $\|u\|_\varepsilon$ is the Sobolev norm of order ε. According to a fundamental theorem of Kohn and Nirenberg [**3**], subelliptic operators are hypoelliptic and satisfy the *a priori* estimates

$$(2) \qquad \|u\|_{s+2\varepsilon}^2 \leqq c_s(\|\mathscr{L}u\|_s^2 + \|u\|^2), \qquad u \in C_0^\infty(V),$$

for each $s \geqq 0$.

In this note we shall display an operator on a Euclidean space which is subelliptic of order $\frac{1}{2}$ at each point and construct an explicit integral operator which inverts it.

2. **Construction of the operator.** Let N be the nilpotent Lie group whose underlying manifold is $\boldsymbol{C}^n \times \boldsymbol{R}$ with coordinates $(z_1, \ldots, z_n, t) = (z, t)$ and whose group law is

$$(z, t)(z', t') = (z + z', t + t' + 2 \operatorname{Im} z \cdot z')$$

where $z \cdot z' = \sum_1^n z_j \bar{z}_j'$. Letting $z = x + iy$, then, $x_1, \ldots, x_n, y_1, \ldots, y_n, t$ are real coordinates on N. We set

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \qquad Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}, \qquad T = \frac{\partial}{\partial t},$$

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right), \qquad \frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right),$$

$$Z_j = \tfrac{1}{2}(X_j - iY_j), \qquad \bar{Z}_j = \tfrac{1}{2}(X_j + iY_j).$$

The following proposition is easily verified.

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