BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 79, Number 1, January 1973

FINITELY GENERATED SUBMODULES OF DIFFERENTIABLE FUNCTIONS. II

BY B. ROTH

Communicated by François Treves, July 24, 1972

1. **Introduction.** Let $\mathscr{E}(\Omega)$ denote the space of real-valued infinitely differentiable functions on an open set Ω in $\mathscr{R}^n$ equipped with the topology of uniform convergence of all derivatives on all compact subsets of Ω. Throughout we assume that Ω is connected.

Let $[\mathscr{E}(\Omega)]^p$ denote the Cartesian product of $\mathscr{E}(\Omega)$ with itself p-times equipped with the product topology. Then $[\mathscr{E}(\Omega)]^p$ is a Frechet space and a $\mathscr{E}(\Omega)$-module. In [3], the finitely generated submodules of $[\mathscr{E}^m(\Omega)]^p$ which are closed in $[\mathscr{E}^m(\Omega)]^p$ are characterized for $m < \infty$ and we are here concerned with the same problem for $m = \infty$.

2. **The main result.** Consider the finitely generated submodule $M = \{g_1 f_1 + \cdots + g_q f_q : g_1, \ldots, g_q \in \mathscr{E}(\Omega)\}$ of $[\mathscr{E}(\Omega)]^p$ where $f_j = (f_{1j}, \ldots, f_{pj}) \in [\mathscr{E}(\Omega)]^p$ for $1 \leqq j \leqq q$. Let F be the $p \times q$ matrix $(f_{ij})_{1 \leqq i \leqq p; 1 \leqq j \leqq q}$. Then $F : [\mathscr{E}(\Omega)]^q \to [\mathscr{E}(\Omega)]^p$ and $\mathrm{im}(F) = M$. In [**2**, pp. 21–25], Malgrange shows that $M = \mathrm{im}(F)$ is closed in $[\mathscr{E}(\Omega)]^p$ if each f_{ij} is real analytic on Ω. A zero of a function is said to be a *zero of finite order* if some derivative of the function fails to vanish there. Our main result is

THEOREM 1. *Suppose* $F = (f_{ij})_{1 \leqq i \leqq p; 1 \leqq j \leqq q}$, $f_{ij} \in \mathscr{E}(\Omega)$, *and let* $r = \max\{\mathrm{rank}(F(x)) : x \in \Omega\}$. *For* $\Omega \subset \mathscr{R}^n$, *if the finitely generated submodule* $\mathrm{im}(F)$ *is closed in* $[\mathscr{E}(\Omega)]^p$, *then for every* $x \in \Omega$ *with* $\mathrm{rank}(F(x)) < r$ *there exists an* $r \times r$ *submatrix* A *of* F *such that* x *is a zero of finite order of* $\det(A)$. *For* $\Omega \subset \mathscr{R}^1$, *the converse also holds.*

For $\Omega \subset \mathscr{R}^n$, $n > 1$, the converse fails to hold [**1**, p. 89]. For $\Omega \subset \mathscr{R}^1$, the fact that the zeros of finite order condition is sufficient follows from Malgrange's characterization of the closure of a submodule of differentiable functions [**1**, Corollary 1.7, p. 25]. For $\Omega \subset \mathscr{R}^n$, the necessity of the zeros of finite order condition can be demonstrated in the following manner. Assuming that $\mathrm{im}(F)$ is closed in $[\mathscr{E}(\Omega)]^p$, we have by the closed range theorem for Frechet spaces that $\mathrm{im}(F') = [\ker(F)]^\perp$ where $F' : [\mathscr{E}'(\Omega)]^p \to [\mathscr{E}'(\Omega)]^q$ is the transpose of F. Assuming that the set Z_∞ of $x \in \Omega$ for which x is a zero of infinite order of $\det(A)$ for every $r \times r$

AMS (MOS) subject classifications (1970). Primary 46E25, 46E40; Secondary 34A30, 35G05, 46F10.

Key words and phrases. Spaces of differentiable functions, modules of differentiable functions, finitely generated submodules, spaces of distributions, systems of linear differential equations.