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ORBITS OF FAMILIES OF VECTOR FIELDS AND INTEGRABILITY OF SYSTEMS WITH SINGULARITIES

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Let X be a C^∞ vector field on the C^∞ manifold M. We use $t \to X_t(m)$ to denote the integral curve of X which passes through m when $t = 0$. Let D be a set of C^∞ vector fields on M. Two points m and m' of M are said to be *D-connected* if there exist elements $X^1, \ldots, X^k$ of D and real numbers $t_1, \ldots, t_k$ such that

$$m' = X^1_{t_1}(X^2_{t_2}(\cdots X^k_{t_k}(m)\cdots)).$$

This defines an equivalence relation on M. The equivalence classes are called the *orbits* of D.

Let S be an orbit of D. For each $m \in S$ and each finite sequence $\xi = (X^1, \ldots, X^k)$ of elements of D, let $F_{\xi,m}$ denote the map

$$(t_1, \ldots, t_k) \to X^1_{t_1}(X^2_{t_2}(\cdots X^k_{t_k}(m)\cdots)).$$

It is clear that $F_{\xi,m}$ is a C^∞ mapping from an open subset U of R^k into M. Moreover the range of $F_{\xi,m}$ is a subset of S. We topologize S by the strongest topology for which all the maps $F_{\xi,m}$ are continuous.

THEOREM 1. *S is a connected C^∞ submanifold of M.*

A *distribution* on M is a mapping H which assigns to every $m \in M$ a linear subspace $H(m)$ of the tangent space of M at m. It is not required that the dimension of $H(m)$ be constant. A vector field X defined in an open subset U of M *belongs* to the distribution H if $X(m) \in H(m)$ for every $m \in U$. We say that H is a C^∞ *distribution* if, for every $m \in M$ and every $v \in H(m)$, there exists a C^∞ vector field X such that X belongs to H and $X(m) = v$. If D is a set of vector fields and H is a distribution, we say that H is D-invariant if, whenever $m \in M$, $X \in D$, and t is a real number such that $X_t(m)$ is defined, it follows that the differential of X_t maps $H(m)$ into $H(X_t(m))$. Given a set D of C^∞ vector fields on M, there exists a smallest distribution H which is D-invariant and is such that every element of D belongs to H. Let this distribution be denoted by P_D. Then P_D is a C^∞ distribution.

Integral manifolds and maximal integral manifolds of a distribution

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