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INJECTIVE MODULES AND CLASSICAL LOCALIZATION IN NOETHERIAN RINGS

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One of the main problems in the growing theory of noncommutative Noetherian rings can be loosely stated thus: If $\mathfrak{p}$ is a prime ideal of a Noetherian ring R, what should one mean by the localization $R_{\mathfrak{p}}$ of R at $\mathfrak{p}$? When does $R_{\mathfrak{p}}$ exist and when is it nice? This problem has been considered by Goldie [**1**] and by Lambek and Michler [**5**]. In this note, we indicate a new approach to this problem and some of its advantages. We also introduce the concept of a left exact biradical for a ring, which may be of independent interest. Details will appear elsewhere.

As usual, a ring is Noetherian if it has the ascending chain condition on right ideals as well as left ideals. A subset of a ring is an Ore set if it is right Ore as well as left Ore. We refer the reader to [**9**] for all unexplained terminology and results concerning left exact radicals.

Let R be a ring. The complete lattice of all left exact radicals for mod-R (resp. R-mod) is denoted as $\boldsymbol{K}_r$ (resp. $\boldsymbol{K}_l$). If $\mathscr{D}$ is a multiplicatively closed subset of R, $\rho_{\mathscr{D}} \in \boldsymbol{K}_r$ and $\lambda_{\mathscr{D}} \in \boldsymbol{K}_l$ are defined as follows: For each $M \in$ mod-R (resp. $M \in R$-mod), $\rho_{\mathscr{D}}(M)$ (resp. $\lambda_{\mathscr{D}}(M)$) is the largest submodule of M, each element of which is annihilated by some element of $\mathscr{D}$. If $\mathfrak{a}$ is an ideal of R, we define $\rho_{\mathfrak{a}}^{\#}$ as $\sup\{\rho \in \boldsymbol{K}_r | \rho(R/\mathfrak{a}) = 0\}$ and $\lambda_{\mathfrak{a}}^{\#}$ as $\sup\{\lambda \in \boldsymbol{K}_l | \lambda(R/\mathfrak{a}) = 0\}$. The multiplicatively closed set $\{r \in R | [r + \mathfrak{a}]$ is regular in $R/\mathfrak{a}\}$ is denoted as $\mathscr{C}(\mathfrak{a})$.

THEOREM 1 (*cf.* [**5**]). *If $\mathfrak{s}$ is a semiprime ideal in a right Noetherian ring then* $\rho_{\mathfrak{s}}^{\#} = \rho_{\mathscr{C}(\mathfrak{s})}$.

Matlis [**6**] has used localization to show that injective modules over a commutative Noetherian ring are nice. In the following two theorems, we establish an intimate connection between localizability and niceness of certain right injectives over a right Noetherian ring. Also see Theorems 7 and 8.

THEOREM 2. *Let $\mathfrak{s}$ be a semiprime ideal in a right Noetherian ring R. Then the following four conditions are equivalent:*

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