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ISOTOPIES OF HOMEOMORPHISMS OF RIEMANN SURFACES AND A THEOREM ABOUT ARTIN'S BRAID GROUP

BY JOAN S. BIRMAN AND HUGH M. HILDEN

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Let $\tilde{X}$, X be orientable surfaces. Let $(p, \tilde{X}, X)$ be a regular covering space, possibly branched, with finitely many branch points and a finite group of covering transformations. We require also that every covering transformation leave the branch points fixed. A homeomorphism $\tilde{g}: \tilde{X} \to \tilde{X}$ is said to be "fiber-preserving" with respect to the triplet $(p, \tilde{X}, X)$ if for every pair of points $\tilde{x}, \tilde{x}' \in \tilde{X}$ the condition $p(\tilde{x}) = p(\tilde{x}')$ implies $p\tilde{g}(\tilde{x}) = p\tilde{g}(\tilde{x}')$. If $\tilde{g}$ is fiber-preserving and isotopic to the identity map via an isotopy $\tilde{g}_s$, then $\tilde{g}$ is said to be "fiber-isotopic to 1" if, for every $s \in [0, 1]$, the homeomorphism $\tilde{g}_s$ is fiber-preserving.

The condition that an isotopy be a fiber-isotopy imposes a symmetry which one feels, intuitively, is very restrictive. However, we find

THEOREM 1. *Let $g: \tilde{X} \to \tilde{X}$ be a fiber-preserving homeomorphism which is isotopic to the identity map. If the covering is branched, assume $\tilde{X}$ is not the closed sphere or torus. Then g is fiber-isotopic to the identity.*

Theorem 2 expresses a weaker result, which is true without exception.

THEOREM 2. *Let $\tilde{g}: \tilde{X} \to \tilde{X}$ be a fiber-preserving homeomorphism which is isotopic to the identity map. Then its projection g to X is also isotopic to the identity map; however, the isotopy may move branch points.*

A special case of Theorem 1 was established by the authors in an earlier paper [1] for the particular situation where X is a 2-sphere, and $\tilde{X}$ is a 2-sheeted covering of X with $2g + 2$ branch points. The proof given here is considerably simpler than the version in [1], and at the same time it holds in a much more general situation. The major tool that made this possible was the device of lifting maps to the universal covering space. The analogous problem in higher-dimensional manifolds has also been studied by the authors, and will be reported on separately.

Let $H(\tilde{X})$ be the group of all orientation-preserving homeomorphisms of $\tilde{X} \to \tilde{X}$, and let $D(\tilde{X})$ be the subgroup of those homeomorphisms which are isotopic to the identity map. Let $M(\tilde{X})$ be the quotient group $H(\tilde{X})/D(\tilde{X})$, that is the mapping class group of $\tilde{X}$. Assume that the

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