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MINIMAL REALIZATION OF MACHINES IN CLOSED CATEGORIES

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There has been recent interest in extending automaton theory to encompass aspects of machine, language, and control theories [**1**], [**2**]. This paper presents a minimal realization theory for discrete-time machines in suitable categories with such applications, but assuming no background outside pure mathematics except perhaps as motivation. Minimal realization is proved in the strong form of an adjunction between behavior and a realization construction generalizing Nerode's [**8**]. We wish to thank Michael Arbib, Lee Carlson, Saunders Mac Lane, and Lotfi Zadeh for their encouragement and/or technical assistance.

If $\boldsymbol{C}$ is a category, $|\boldsymbol{C}|$ denotes its class of objects; and the composite of $A \xrightarrow{f} B \xrightarrow{g} C$ is written $A \xrightarrow{fg} C$. Functors are written after their arguments.

1. X-**automata**. In the first four sections $\boldsymbol{C}$ is a fixed *suitable* category, i.e., closed symmetric monoidal [**7**] with countable coproducts and canoical cofactorizations [**6**]. Let ***Mon*** be the category of *monoids* [**7**] in $\boldsymbol{C}$. For $X \in |\boldsymbol{C}|$, let $X^* = \coprod_t \otimes^t X$, the coproduct over the nonnegative integers of iterated "tensor" powers $\otimes^t$ of X, where $\otimes$ is the multiplication in $\boldsymbol{C}$. Let $i_0 : I \to X^*$ be the zeroth injection, from $\otimes^0 X = I$, the identity for $\otimes$ in $\boldsymbol{C}$. Finally, define $\mu : X^* \otimes X^* \to X^*$ to be the composite

$$\left(\coprod_r \otimes^r X\right) \otimes \left(\coprod_s \otimes^s X\right) \cong \coprod_{r,s} (\otimes^r X) \otimes (\otimes^s X) \cong \coprod_{r,s} \otimes^{r+s} X \to X^*$$

where the first isomorphism uses the distributivity of $\otimes$ over $\coprod$ which arises from the adjointness of $\otimes$, the second isomorphism is a generalized associative law in $\boldsymbol{C}$, and the third morphism is defined by letting its $\langle r, s\rangle$-component be the $r + s$ injection $\otimes^{r+s} X \to X^*$. Then [**7**], $\langle X^*, \mu, i_0\rangle \in |\boldsymbol{Mon}|$.

For $M \in |\boldsymbol{Mon}|$, let $\boldsymbol{Act}^M$ be the category of right M-actions in $\boldsymbol{C}$, that is, $\alpha : S \otimes M \to S$ satisfying appropriate identities [**7**]. For $X \in |\boldsymbol{C}|$, an X-*monadic algebra* in $\boldsymbol{C}$ is $\delta : S \otimes X \to S$; and a morphism $h : \delta \to \delta'$ of such algebras is $h : S \to S'$ such that $(h \otimes X)\delta' = \delta h$. Let $\boldsymbol{Mond}^X$ be the resulting category. δ is often called a *transition* morphism.

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