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A SPECTRAL MAPPING THEOREM FOR TENSOR PRODUCTS OF UNBOUNDED OPERATORS

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1. **Introduction.** In this note we will discuss the spectrum of tensor products of not necessarily bounded operators on Banach spaces X and Y. $X \otimes Y$ will denote the tensor product of X and Y in some uniform cross-norm [1]. Thus, (i) $X \otimes Y$ is the completion of the algebraic tensor product in a norm with $\|x \otimes y\| = \|x\| \, \|y\|$; (ii) for any $A \in \mathscr{L}(X)$, the bounded operators on X, and $B \in \mathscr{L}(Y)$, there is an operator $A \otimes B \in \mathscr{L}(X \otimes Y)$ with $(A \otimes B)(x \otimes y) = Ax \otimes By$ and $\|A \otimes B\| = \|A\| \, \|B\|$. Typical examples of such uniform cross-norms are the usual Hilbert space tensor product norm and the L^p norm on $L^p(X \otimes Y, d\mu \otimes d\nu) = L^p(X, d\mu) \otimes L^p(Y, d\nu)(1 \leqq p < \infty)$.

Given a polynomial (or a rational function) in two variables and closed operators A on X and B on Y, we want to discuss the spectrum of $P(A \otimes I, I \otimes B)$ as an operator on $X \otimes Y$. For unbounded operators, one must define what it means for an operator C on $X \otimes Y$ "to be" $P(A \otimes I, I \otimes B)$. We take a fairly strong definition:

DEFINITION 1. Given a closed operator A with nonempty resolvent set on a Banach space, X, we say that a sequence A_n of bounded operators on X is an *$\mathscr{R}(A)$-approximation* if and only if A_n converges to A in norm resolvent sense [2] and each A_n is a polynomial in resolvents of A.

DEFINITION 2. Given closed operators A and B on Banach spaces X and Y, and a rational function, $P(z, \omega)$, we say that a closed operator C on $X \otimes Y$ *equals* $P(A \otimes I, I \otimes B)$ (or $P(A, B)$, for short) if and only if, there exists an $\mathscr{R}(A)$-approximation, A_n, and an $\mathscr{R}(B)$-approximation, B_n, so that $P(A_n, B_n)$ converges in norm resolvent sense to C.

Existence and uniqueness questions for $P(A, B)$ naturally arise. In applying Theorem 1 below, all the hard analysis is in proving that existence holds. The existence and uniqueness question is discussed in detail in a forthcoming paper [3], primarily in the case where A and B are generators of bounded holomorphic semigroups. In the general case, we do not know whether it is possible for two different operators C and C' to both "equal" $P(A, B)$ but in that case our proof of Theorem 1 implies that $(C - \lambda)^{-1} - (C' - \lambda)^{-1}$ is quasinilpotent.

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