BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 78, Number 5, September 1972

THE DIFFERENTIAL CLOSURE OF A DIFFERENTIAL FIELD

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Good afternoon ladies and gentlemen. The subject of mathematical logic splits fourfold into: recursive functions, the heart of the subject; proof theory which includes the best theorem in the subject; sets and classes, whose romantic appeal far outweigh their mathematical substance; and model theory, whose value is its applicability to, and roots in, algebra. This afternoon I hope to sketch some theorems about differential fields first derived by model theoretic methods. In particular, I will indicate why every differential field $\mathscr{A}$ of characteristic 0 has a unique prime differentially closed extension called the differential closure of $\mathscr{A}$. Model theory has proved useful in the study of differential fields because the notion of differential closure is surprisingly more complex than the analogous notions of algebraic closure, real closure, or Henselization. The virtue of model theory is its ability to organize succinctly the sort of tiresome algebraic details associated with elimination theory.

The first concepts of model theory are structure and theory. Typic structures are groups, rings and fields. A theory is a set of sentences. A sentence is about the elements of some structure[2]. The language of fields includes plus (+), times (·), equals (=) and variables that stand for elements of fields. A typic sentence in the language of fields says: every polynomial of degree 7 has a root. A typic theory is the theory of algebraically closed fields of characteristic 0 (ACF_0). A structure $\mathscr{A}$ is said to be a model of a theory T if every sentence of T is true in $\mathscr{A}$. Thus the models of ACF_0 are what men call algebraically closed fields of characteristic 0.

Pure model theory, at first thought, appears to be too general to have any mathematical substance. But that hasty thought is given the lie by several theorems, one of which is due to Vaught [1]: Let T be any coun-

AMS 1970 *subject classifications*. Primary 02H15; Secondary 12H05.
Key words and phrases. Differentially closed fields, uniqueness of differential closure, totally transcendental theories, model completion, Robinson's principle.

[1] What follows is a verbatim rendering of an invited address given by the author to the 688th meeting of the American Mathematical Society in Cambridge, Massachusetts on October 30, 1971, verbatim save for the deletion of some improprieties that seemed ornamental on first hearing but proved meretricious in cold print. The author wishes to thank Doctor Lenore Blum for teaching him the essentials of differential fields. Preparation of this paper was partially supported by NSF Grant GP-29079; received by the editors January 13, 1972.

[2] I speak here of first order sentences. Second order sentences are about subsets of structures and belong more to analysis than to algebra. True, ideals are subsets rather than elements, but they tend to be finitely generated and consequently first order in nature.