BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 78, Number 4, July 1972

A GENERALIZED MÖBIUS INVERSION FORMULA

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Communicated by G.-C. Rota, December 27, 1971

1. **Introduction.** We owe to G.-C. Rota [*On the foundations of combinatorial theory* I. *Theory of Möbius functions*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete, Band 2, (1964), 340–368] the idea of obtaining explicit formulas for the chromatic polynomials of graphs or maps by use of a so-called Möbius inversion formula. His formulas give the development of these polynomials in powers of λ, where λ is the number of colors. His work provides interesting commentary on previous work of G. D. Birkhoff [*A determinant formula for the number of ways of coloring a map*, Ann. of Math., (2) **14** (1912), 42–46] and H. Whitney [*A logical expansion in mathematics*, Bull. Amer. Math. Soc. **38** (1932), 572–579] who had discovered similar formulas by methods which, at least superficially, seemed quite different.

Now, in the case of regular planar maps, the developments in powers of $(\lambda - 2)$ or $(\lambda - 3)$ appear to have great advantages over the developments in powers of λ. General formulas for developments in powers of $(\lambda - 3)$ have so far eluded us. But general formulas for the coefficients, $a_1, a_2, \ldots$, in expansions for chromatic polynomials in the form

$$\lambda(\lambda - 1) \sum_{k=1}^{n-2} a_k(\lambda - 2)^k$$

are well known. They are given in a paper by G. D. Birkhoff and D. C. Lewis [*Chromatic polynomials*, Trans. Amer. Math. Soc., **60** (1946), 355–451], hereafter referred to as BL. The question arose whether these formulas could be obtained by use of a Möbius function for a suitably chosen partially ordered set. It seems impossible to do so by just using the Möbius function in the form given by Rota.

The purpose of this paper is to present an extremely simple generalization of the Rota-Möbius inversion formula which suffices to prove the so-called determinant formula for the chromatic polynomial for a regular planar map as given in BL, pp. 401–405, in powers of $(\lambda - 2)$. In §4, we actually give the new proof of the determinant formula, omitting the details about the so-called markings of the maps, which are supplied by appropriate passages in BL.

2. **The principal theorem.** Let S be a locally finite partially ordered set with elements $x, y, z, \ldots$. Let S_x be a finite subset of S, defined for each

AMS 1970 *subject classifications*. Primary 05C15, 06A10.