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# HECKE RINGS OF CONGRUENCE SUBGROUPS

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Let $k$ be a $\mathfrak{p}$-adic field and let $\hat{G}$ be a reductive group defined over $k$. Let $G$ be a semigroup in $\hat{G}$, i.e. a multiplicative subset with the same unity as $\hat{G}$ We shall assume that there exists an open compact subgroup $\Delta$ of $\hat{G}$ which is contained in $G$. Let $\mathscr{R}(G, \Delta)$ be the free $Z$-module generated by the double cosets of $G$ modulo $\Delta$, with a product defined as in [3, Lemma 6]. We have an associative ring with unity which we shall call the Hecke Ring of $G$ with respect to $\Delta$. Let $\Delta_0$ be a normal subgroup of $\Delta$ satisfying our conditions H-1 and H-2 of §1. Our purpose is to find generators and relations for $\mathscr{R}(G, \Delta_0) = \mathscr{R}$. There exists a finitely generated polynomial ring $Z[G]$ which together with the group ring $Z[\Delta/\Delta_0]$ generates $\mathscr{R}$; moreover $\mathscr{R}$ is a $Z[\Delta/\Delta_0]$-bimodule having $Z[D]$ as basis. Our hypothesis H-1 and H-2 are verified for the principal congruence subgroups of most of the classical groups considered in [2].

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1. **General results.** Let $T$ be a connected $k$-closed subgroup of $G$ consisting only of semisimple elements, and $N^+$ and $N^-$ be maximal $k$-closed unipotent subgroups normalized by $T$. We set $N^+ = N^+ \cap \Delta$, and $U^- = N^- \cap \Delta$. We shall now state our first condition:

*Condition* H-1. There exists a finitely generated semigroup $D$ in $T$ such that $G = \Delta D \Delta$ (disjoint union), and for all $d \in D$ we have $dU^+d^{-1} \subset U^+$ and $d^{-1}U^-d \subset U^-$.

We turn now to our second condition. We let $\Delta_0$ be a normal subgroup of $\Delta$ and we set $U_0^+ = U^+ \cap \Delta_0$ and $U_0^- = U^- \cap \Delta_0$. We shall assume that $T \cdot N^+ \cap \Delta_0 = (T \cap \Delta_0) \cdot U_0^+$.

*Condition* H-2. There exists a semigroup $D$ in $T$ such that $\Delta_0 = U_0^+ V U_0^-$ for a certain subgroup $V$ of $\Delta_0$ normalized by $D$, and for all $d$ in $D$ we have $dU_0^+d^{-1} \subset U_0^+$ and $d^{-1}U_0^-d \subset U_0^-$.

Let us denote by $\bar{1}$ the unity of $\mathscr{R}$ and by $\bar{g}$ the double coset $\Delta_0 g \Delta_0$. We shall denote the product in $\mathscr{R}$ by $*$.

THEOREM 1. *Condition* H-2 *implies that* $D = \Delta_0 D \Delta_0$ *is a semigroup in* $\hat{G}$ *and* $\mathscr{R}(\hat{D}, \Delta_0) \simeq Z[D]$.

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