BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 78, Number 4, July 1972

LATTICES OF INVARIANT SUBSPACES

BY WILLIAM ARVESON[1]

Communicated by Irving Glicksberg, December 27, 1971

Let $\mathscr{A}$ be an algebra of operators on a Hilbert space $\mathscr{H}$, and let Lat $\mathscr{A}$ denote the lattice of all $\mathscr{A}$-invariant subspaces. Recall [4] that $\mathscr{A}$ is said to be *reflexive* if every operator which leaves invariant every element of Lat $\mathscr{A}$ is already in $\mathscr{A}$. Thus, reflexive operator algebras are completely determined by their invariant subspace lattices, and one might hope to get useful information about the algebra $\mathscr{A}$ by studying the lattice Lat $\mathscr{A}$. One question that arises here is, which specific lattices occur as the lattice Lat $\mathscr{A}$ of some algebra $\mathscr{A}$? Such lattices are called *reflexive* (it is very easy to see that reflexive lattices have an equivalent definition which is dual to the definition of reflexive algebra). In this paper we announce some results of a study of reflexive lattices; full details will appear elsewhere.

Before going further, we want to describe the type of concrete problems which led to these considerations. To specify a reflexive algebra $\mathscr{A}$ on a separable space $\mathscr{H}$, it is equivalent to specify a sequence $\mathfrak{M}_1, \mathfrak{M}_2, \ldots$ of closed subspaces of $\mathscr{H}$ ($\mathscr{A}$ is then defined as $\{T: T\mathfrak{M}_i \subseteq \mathfrak{M}_i, i \geqq 1\}$). To get a fairly general class of examples, let n be a positive integer, and let X be the countable Cartesian product $X_1 \times X_2 \times \cdots$ of the sets $X_i = \{1, 2, \ldots, n\}$. X comes equipped with a product Borel structure, and we define subsets $E_{ij} \subseteq X$, $i \geqq 1$, $1 \leqq j \leqq n$, to be the cylinders

$$E_{ij} = \{x \in X : x_i \leqq j\},$$

where x_i denotes the ith coordinate of the sequence x. Now choose a finite measure μ on X, let $\mathscr{H} = L^2(X, \mu)$, and define subspaces $\mathfrak{M}_{ij}$ of $\mathscr{H}$ by $\mathfrak{M}_{ij} = L^2(E_{ij}, \mu)$. Then the operator algebra

$$\mathscr{A}(X, \mu) = \{T \in \mathscr{L}(\mathscr{H}) : T\mathfrak{M}_{ij} \subseteq \mathfrak{M}_{ij}, \text{for all } i, j\}$$

is reflexive, and it always contains the multiplication algebra of $L^\infty(X, \mu)$. Of course, the properties of these algebras will depend strongly on the measure μ. We want to take up two types of special cases:

Problem 1. Let $n \geqq 2$, and define μ on X by $\mu = \prod_1^\infty \mu_i$, where each μ_i assigns uniform mass $1/n$ to each point of $\{1, 2, \ldots, n\}$. Let $\mathscr{A}_n$ be the associated operator algebra. Can $\mathscr{A}_m$ and $\mathscr{A}_n$ be similar if $m \neq n$?

Problem 2. Take $n = 2$, and choose a number p, $0 < p < 1$. Let $\mu = \prod_1^\infty \mu_i$, where each μ_i assigns mass p to $\{1\}$ and mass $1 - p$ to $\{2\}$. Let $\mathscr{A}_p$ be the associated operator algebra. Can $\mathscr{A}_p$ and $\mathscr{A}_q$ be similar if $p \neq q$?

AMS 1970 *subject classifications*. Primary 41A15; Secondary 46L15.

[1] Research supported by NSF grant GP-5585.