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GENERALIZED RAMSEY THEORY FOR GRAPHS

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The classical Ramsey numbers [7] involve the occurrence of monochromatic complete subgraphs in line-colored complete graphs. By removing the completeness requirements and admitting arbitrary forbidden subgraphs within any given graph, the situation is richly and nontrivially generalized.

The Ramsey number $r(m, n)$ as traditionally studied in graph theory [5, p. 15] may be defined as the minimum number p such that every graph with p points which does not contain the complete graph K_m must have n independent points. Alternatively, it is the smallest p for which every coloring of the lines of K_p with two colors, green and red, contains either a green K_m or a red K_n. Thus the *diagonal Ramsey numbers* $r(n, n)$ can be described in terms of 2-coloring the lines of K_p and regarding K_n as a forbidden monochromatic subgraph without regard to color.

This viewpoint suggests the more general situation in which an arbitrary graph G has a c-coloring of its lines and the number of monochromatic occurrences of a forbidden subgraph F (or of a forbidden family of graphs) is calculated. A host of problem areas within graph theory can be subsumed under such a formulation. These include the line-chromatic number, in which the 3-point path is forbidden. The arboricity of G involves forbidding all cycles. The thickness of a graph forbids the Kuratowski graphs. Complete bipartite graphs can be taken for both G and F, and so can cubes Q_n and Q_m.

There has long been a sentiment in graph theory that there is an intimate relationship between extremal graph theory and Ramsey numbers. It does not appear possible to derive either Turán's theorem or Ramsey's theorem from the other. However, we show that extremal bipartite graph theory does in fact imply the bipartite form of Ramsey's theorem.

Let $\mathscr{F}$ be a family of graphs, G a given graph, and c a positive integer. We denote by $R(G, \mathscr{F}, c)$ the greatest integer n with the property that, in every coloration of the lines of G with c colors, there are at least n monochromatic occurrences of a member of $\mathscr{F}$. Without any loss of generality, we can assume that every forbidden subgraph $F \in \mathscr{F}$ has no isolated points. If $\mathscr{F}$ contains just one forbidden subgraph F then we write simply $R(G, F, c)$ instead of $R(G, \{F\}, c)$.

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