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SURFACES IN CONSTANT CURVATURE MANIFOLDS WITH PARALLEL MEAN CURVATURE VECTOR FIELD

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I. **Statement of results.** For an (n)-dimensional Riemannian manifold M^n, isometrically immersed in an $(n+k)$-dimensional Riemannian manifold $M_{(c)}^{(n+k)}$ of constant sectional curvature c, let H denote the mean curvature vector field of M^n. H is a section of the normal bundle NM^n of the immersion. When $n = 2$, $k = 1$, and $c = 0$ (a surface immersed in E^3), the requirement $|H| =$ constant is classical constant mean curvature. If $k > 1$, however, the condition $|H| =$ constant is usually too weak to prove reasonable generalizations of the classical theorems for surfaces of constant mean curvature in E^3. We investigate a stronger requirement on H; namely, that H be parallel with respect to the induced connection in the normal bundle (for definitions, see II). Then using an analytic construction first employed by H. Hopf [**2**], we obtain

THEOREM 1. *A compact surface M^2 of genus* 0 *immersed in $M^4(c)$, $c \geqq 0$, upon which H is parallel in the normal bundle, is a sphere of radius* $1/|H|$.

This generalizes a theorem of Hopf, who proved that the only immersed surfaces in E^3 of genus 0 with constant mean curvature are spheres [**2**, Chapter 7, §4]. For complete surfaces in E^4, we prove

THEOREM 2. *A complete surface M^2, immersed in E^4, whose Gauss curvature does not change sign, and for which H is parallel in the normal bundle NM^2, is a minimal surface ($H \equiv 0$), a sphere, a right circular cylinder, or a product of circles $S^1(r) \times S^1(\rho)$, where $|H| = \frac{1}{2}(1/r^2 + 1/\rho^2)^{1/2}$.*

This extends a theorem of Klotz and Osserman for complete surfaces of constant scalar mean curvature in E^3 [**5**]. It can also be generalized to immersions into $\overline{M}_{(c)}^4$, $c \geqq 0$. Theorem 2 is proved in two steps. First we prove

THEOREM 3. *A piece of immersed surface M^2 in E^4, satisfying the conditions of Theorem* 2 *with $H \neq 0$, is either a sphere or it is flat* ($K = 0$).

Then we establish the following characterization of flat surfaces in E^4 with parallel mean curvature vector fields:

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