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ANALYTIC FUNCTIONS WITH UNIVALENT DERIVATIVES AND ENTIRE FUNCTIONS OF EXPONENTIAL TYPE

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ABSTRACT. Functions f, analytic and univalent in the unit disc, and such that all successive derivatives $f^{(k)}$ are univalent in this disc, are necessarily transcendental entire functions of exponential type. These functions, and functions f having an infinite number of derivatives $f^{(n_k)}$ univalent in the unit disc, are discussed. Entire functions of bounded index are of exponential type and their properties are also discussed.

1. **Introduction.** Let $f(z)$ be analytic in the unit disc $D:|z| < 1$. We say that f is univalent in D if for each pair of distinct points z_1, z_2 in D, $f(z_1) \neq f(z_2)$. In §§1–4 we give a brief survey of functions analytic and[1] univalent in D. Functions f such that $f(z)$ and each successive derivative $f^{(k)}(z)$ are univalent in D are considered next in §5. Such functions f must be transcendental entire functions of exponential type. Related problems of functions f such that $f(z)$ and a sequence of derivatives $f^{(n_k)}(z)$ are univalent or of functions f such that $f(z)$ is entire and $f^{(k)}(z)$ is univalent in $|z| < \rho_k$ $(\rho_k > 0)$ are considered in §§6–10. This is followed by a section (§11) on multivalent functions and three sections (§§12–14) on functions of bounded index. An entire function $f(z)$ is said to be of bounded index if there exists an integer N, independent of z, such that

$$\max_{0 \leqq s \leqq N} \left\{ \frac{|f^{(s)}(z)|}{s!} \right\} \geqq \frac{|f^{(j)}(z)|}{j!}, \tag{1.1}$$

for $j = 1, 2, \ldots$ and for all z. The smallest such integer N is called the index of f. An entire function f of bounded index N is of exponential type not exceeding $(N + 1)$. Finally we mention some unsolved problems.

2. **Conditions for the univalence of f.** Let

$$f(z) = \sum_0^\infty a_n z^n, \qquad |z| < 1. \tag{2.1}$$

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[1] In this article we shall not consider meromorphic univalent functions.