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SIGNATURES ON SEMILOCAL RINGS

BY MANFRED KNEBUSCH, ALEX ROSENBERG[1] AND ROGER WARE[2]

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We announce extensions of a part of the Artin-Schreier theory of real fields to semilocal rings. Detailed proofs will appear elsewhere.

A always denotes a (not necessarily noetherian) semilocal commutative ring such that no residue class field has only two elements. A *signature* on A is a homomorphism σ from the unit group, A^*, of A to $\{\pm 1\}$ with $\sigma(-1) = -1$ and $\sigma(l^2 + am^2) = 1$ for all triples (a, l, m) in $A^* \times A \times A$ such that $l^2 + am^2$ is a unit and $\sigma(a) = 1$.

EXAMPLES. (i) If A is an integral domain and $<$ is a total ordering of A then $\sigma: A^* \to \{\pm 1\}$ defined by $\sigma(a) = 1$ if $a > 0$ and $\sigma(a) = -1$ if $a < 0$ is a signature. If A is a field the signatures correspond bijectively with the orderings of A.

(ii) Let A be the local ring of the affine curve $X^2 + Y^2 = 0$ over the real field $\boldsymbol{R}$ at $(0, 0)$. Then the signature

$$\sigma: A^* \xrightarrow{v} \boldsymbol{R}^* \xrightarrow{s} \{\pm 1\}$$

obtained by composing the evaluation map v at $(0, 0)$ and the unique signature s of $\boldsymbol{R}$ does not arise from an ordering of A.

(iii) For valuation rings we are able to analyze the situation to some extent: Let A be a valuation ring with maximal ideal $\mathfrak{m}$. Then any signature σ arises from an ordering of A. If A has rank one and $\sigma(1 + \mathfrak{m}) \neq 1$, then σ arises from a unique ordering. If A is a discrete rank one valuation ring and $\sigma(1 + \mathfrak{m}) = 1$, then there are exactly two orderings on A inducing the signature σ.

Let A now again be a general semilocal ring as above.

PROPOSITION 1. *Let σ be a signature, $a_1, \ldots, a_r$ units of A, and $l_1, \ldots, l_r$ elements of A such that $b = l_1^2 a_1 + \cdots + l_r^2 a_r$ is also a unit. Then $\sigma(a_1) = \cdots = \sigma(a_r) = 1$ implies $\sigma(b) = 1$.*

DEFINITION. A subset M of A^* is saturated if M is a subgroup of A^* and $a_1, \ldots, a_r$ in M implies b in M for all units $b = l_1^2 a_1 + \cdots + l_r^2 a_r$ with l_i in A.

Thus Proposition 1 states that for any signature σ the set

$$\Gamma(\sigma) = \{a \text{ in } A^* \mid \sigma(a) = 1\}$$

is saturated.

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