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ON A CLASS OF BEST NONLINEAR APPROXIMATION PROBLEMS

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Let $K(\xi, \eta)$ be a real valued differentiable function defined on $T \times T$ where T is an interval of the real line. More exact requirements on $K(\xi, \eta)$ and T will be specified later. In this paper we report results on existence, uniqueness and characterizations of the best approximation in the L^p norm $(1 \leqq p \leqq \infty)$ to functions of the kind

$$h(\xi) = \int_\alpha^\beta K(\xi, \eta)\omega(\eta)\, d\eta, \qquad (\alpha, \beta) \subset T, \tag{1}$$

where $\omega(\eta)$ is positive and continuous, by quadrature functions of the form

$$Q(\xi) = \sum_{i=1}^r c_i K(\xi, \eta_i). \tag{2}$$

Here r is fixed, but c_i real and $\eta_i \in T$ are free parameters. The η_i points are called the knots of the approximating functional. More general expressions of the approximation function allow multiple knots and also boundary terms. Specifically, in place of (2), we consider

$$P(\xi) = \sum_{i=0}^{n-1} a_i \frac{\partial^i}{\partial \eta^i} K(\xi, \alpha) + \sum_{i=0}^{m-1} b_i \frac{\partial^i}{\partial \eta^i} K(\xi, \beta) + \sum_{k=1}^{t} \sum_{l=0}^{\mu_k - 1} c_{kl} \frac{\partial^l K(\xi, \eta_k)}{\partial \eta^{(l)}} \tag{3}$$

where the knots α, β and $\eta_k \in T$ are of multiplicity n, m and μ_k respectively, and the total multiplicity of the interior knots is stipulated to be $\sum_{i=1}^t \mu_i = r$. The functions (2) display each η_i as a simple knot with the terms involving the knots α and β omitted.

The class of functions of the form (3) are designated as $\mathscr{S}_{n,m,r}$.

Our main objective is to characterize the best approximation to $h(\xi)$ in the $L^p(T)$ norm from among the functions in $\mathscr{S}_{n,m,r}$. Formally stated, we wish to establish criteria for evaluating $\{a_i\}$, $\{b_i\}$, $\{c_i\}$ and $\{\eta_i\}$ achieving

$$\min_{a_i, b_i, c_i, \eta_i} \int_T \left| \int_\alpha^\beta K(\xi, \eta)\omega(\eta)\, d\eta - P(\xi) \right|^p d\xi \tag{4}$$

for P of the form (3). We will also investigate the problem of the min

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