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# ON EXTENSIONS OF FUNDAMENTAL GROUPS OF SURFACES AND RELATED GROUPS

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1. The aim of this paper is to sketch a proof of a conjecture of A. Karrass and D. Solitar raised at the American Mathematical Society meeting in Urbana, October 1970: *A finite torsionfree extension of the fundamental group of a closed surface is isomorphic to the fundamental group of a closed surface.* By using the same methods we can prove a similar result for fundamental groups of Seifert fibre spaces.

It is natural that the proofs are quite different for the (euclidean) cases of the torus and Klein bottle and the other (noneuclidean) surfaces. The proof in the euclidean case could be extracted from results on crystallographic groups of Bieberbach, Burckhardt and Frobenius, but I shall sketch an elementary proof here. In the proof for the noneuclidean cases we shall use the theorem of J. Nielsen and S. Kravetz that any finite subgroup of the outer automorphism group of the fundamental group of a closed surface can be realized by a finite group of homeomorphisms [1].[1]

2. THEOREM 1. *Let $\mathfrak{G}$ be a group without torsion, $\mathfrak{F}$ a subgroup of finite index isomorphic to the fundamental group of a closed "noneuclidean" orientable surface (i.e. the genus is $>1$ in the orientable case, $>2$ in the nonorientable). Then $\mathfrak{G}$ is isomorphic to the fundamental group of a closed noneuclidean surface.*

From Kneser's formula [6, p. 50] it follows

COROLLARY. *If $\mathfrak{G}$ and $\mathfrak{F}$ are the fundamental groups of orientable surfaces of genus g and f, resp., and if $\mathfrak{F}$ is a subgroup of $\mathfrak{G}$ with $c=[\mathfrak{G}:\mathfrak{F}]$, then $g-1=c(f-1)$.*

In the proof of Theorem 1 we may restrict ourselves to the case

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[1] ADDED IN PROOF. Recently it was found out that the theorem about negative curvature of the Teichmüller space in [1] is not true. Therefore [1, Theorem 5.2] is not known to be valid and the Theorems 1, 1′ and 5 of this article are still open. In the special cases of finite extensions by solvable groups the theorems will follow from the fixed point theorem of P. A. Smith (Ann. of Math. **35** (1934), 572–578).