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VARIATIONAL PROBLEMS WITHIN THE CLASS OF SOLUTIONS OF A PARTIAL DIFFERENTIAL EQUATION

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1. **Introduction.** A classical problem in the calculus of variations is the optimization of a multiple integral over a domain G of a function containing as arguments the independent variables, the unknown function and its partial derivatives up to order l. Usually the unknown function is required to be an element of the class of all functions that are $2l$-times continuously differentiable defined on an open domain containing G.

The optimization problem that is dealt with in this paper differs from the one above in that the class of admissible functions to be considered is the collection of all sufficiently smooth solutions in G of a given partial differential equation of order greater than or equal to $2l$.

This paper contains the definition of the variational adjoint, a special form of the variational adjoint boundary conditions, and necessary conditions for the elliptic as well as for the parabolic case. The necessary conditions take the form of a boundary value problem.

A physical application occurs in the control with boundary and initial conditions of a process in G that is described by a specific partial differential equation. If the differential equation is of elliptic type the process may be controlled by Dirichlet boundary conditions or by any other set of boundary conditions that generate a class of admissible functions.

2. **Notation and definitions.** R^ν is the ν-dimensional Euclidean space. G is an open bounded domain in R^ν with boundary ∂G. $\partial G \in C^k$ denotes that ∂G is k-times continuously differentiable. If $\partial G \in C^1$ then $\boldsymbol{n}$ is the outward unit normal vector to ∂G. If $A \subset R^\nu$, then nbh A is an open set in R^ν that contains A. If $u \in C^k(G)$, then $\|u(x)\|_k$ denotes the sum of the supremums in G of the absolute values of the function u and all its derivatives of order less or equal to k. If $\alpha \in [1, \nu]$ (α an integer) then $D_\alpha u(x)$ denotes $\partial u(x)/\partial x_\alpha$, $u_{,\alpha}$ is the same as $D_\alpha u(x)$. If $\partial G \in C^k$ and if $u \in C^k(\text{nbh } \partial G)$ then $(\partial/\partial \boldsymbol{n})^k u(x)$ is the kth derivative of u along $\boldsymbol{n}$ on ∂G.

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