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AN EXISTENCE THEOREM FOR ORDINARY DIFFERENTIAL EQUATIONS IN BANACH SPACES[1]

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ABSTRACT. We consider nonlinear ordinary differential equations in Banach spaces. A local existence theorem for the Cauchy problem is given when the equation is continuous in the weak topology. The theorem can be extended to set-valued differential equations in Banach spaces.

Let B be a Banach space and let $F:(0, 1)\times B\to B$. If B is finite dimensional and F is continuous in a neighborhood of $(t_0, x_0)\in(0, 1)\times B$, then by the Peano existence theorem there exists a function $\phi(t)$ defined on a subinterval of $(0, 1)$ such that

$$\phi'(t) = F(t, \phi(t)) \quad \text{and} \quad \phi(t_0) = x_0.$$

Dieudonné [1] and Yorke [2] have shown, by means of examples, that continuity alone, of the function F, is not sufficient to prove a local existence theorem in the case where B is infinite dimensional. Other authors, for example [3] and [4], have extended the Peano theorem to infinite-dimensional spaces but with additional assumptions. We have found that by replacing strong continuity with weak continuity and assuming the range of F to be bounded we may obtain an existence theorem.

Let B be a separable reflexive Banach space with norm $\|\cdot\|$ and let B^* be its dual space. Let B_w denote the space B with the weak topology and let $\{f_i\}$ be a countable dense subset in B^*. By Δ we mean a subinterval of $T=(0, 1)$.

DEFINITION 1. A function $F: T\times B\to B$ is said to satisfy condition (I) if, at each $(t_0, x_0)\in T\times B$,

$$F(t_0, x_0) = \bigcap_{N=1}^{\infty} \operatorname{cl} \operatorname{co} F_N(t_0, x_0)$$

where

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