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NORMAL CONTROL PROBLEMS HAVE NO MINIMIZING STRICTLY ORIGINAL SOLUTIONS[1]

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ABSTRACT. We prove for a general optimal control problem that, in the absence of abnormal admissible extremals (solutions of a generalized Weierstrass E-condition), any control which is optimal in the set of original (ordinary) controls must also be optimal in the larger set of relaxed (measure-valued) controls.

1. We consider the model of an optimal control problem studied in [2]. This model was found applicable, among others, to unilateral control problems defined by ordinary differential and multidimensional integral equations [3], evasion problems [4], and conflicting control problems [5]. For the sake of completeness, we begin by restating the definition of this model. Let T and R be compact metric spaces and μ a positive and nonatomic Radon measure on T. We denote by $\text{rpm}(R)$ the set of regular Borel probability measures on R endowed with the relative weak star topology of $C(R)^*$, by $\mathcal{R}$ the class of μ-measurable functions on T to R (*original control functions*), and by $\mathcal{S}$ the set of μ-measurable functions on T to $\text{rpm}(R)$ (*relaxed control functions*). We embed R in $\text{rpm}(R)$ and $\mathcal{R}$ in $\mathcal{S}$ by identifying each $r \in R$ with the Dirac measure at r. In turn, we view $\mathcal{S}$ as a subset of $L^1(T, C(R))^*$, and endow it with the relative weak star topology, by identifying each $\sigma \in \mathcal{S}$ with the functional $\phi \to \int \mu(dt) \int \phi(t)(r)\sigma(t)(dr)$.

Now let $\boldsymbol{R}$ be the real line, $\mathcal{X}$ a real topological vector space, C a convex body in $\mathcal{X}$, B a convex subset of a vector space (the *set of control parameters*), m a positive integer, $x = (x_0, x_1, x_2): \mathcal{S} \times B \to \boldsymbol{R} \times \boldsymbol{R}^m \times \mathcal{X}$ a given function, and

$$\mathcal{A}(\mathcal{U}) = \{(\sigma, b) \in \mathcal{U} \times B \mid x_1(\sigma, b) = 0, x_2(\sigma, b) \in C\} \qquad (\mathcal{U} \subset \mathcal{S}).$$

We say that $(\bar{\sigma}, \bar{b})$ is a *minimizing original* (respectively *relaxed*) *solution* if it yields a minimum of x_0 on $\mathcal{A}(\mathcal{R})$ (respectively on $\mathcal{A}(\mathcal{S})$). A minimizing original solution is a *minimizing strictly original solution* if it is not at the same time a minimizing relaxed solution. We set $Q = \mathcal{S} \times B$, denote by $\mathfrak{I}_{m+1}$ the simplex $\{(\theta^0, \cdots, \theta^m) \in \boldsymbol{R}^{m+1} \mid \theta^j \geqq 0,$

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