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THE FACTORIZATION PROBLEM FOR NONNEGATIVE OPERATOR VALUED FUNCTIONS

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Introduction. Let F be a function defined on the circle $\Gamma = \{e^{i\theta}: 0 \leqq \theta < 2\pi\}$ or line $R = (-\infty, \infty)$ whose values are nonnegative operators on a separable complex Hilbert space. We are concerned with the problem of finding conditions that $F = G^*G$ a.e. where G is the strong boundary value function of a suitable operator valued analytic function defined in the disk $|z| < 1$ or half-plane $y > 0$. Mainly we are interested in special classes of functions in which such a factorization is always possible.

Our study is motivated by the Fejér-Riesz theorem on the factorization of nonnegative trigonometric polynomials, and Ahiezer's version [1] of its generalization to entire functions of exponential type which are nonnegative on the real axis. Both results generalize to operator valued functions, and, in fact, both appear as special cases of a very general result (Theorem 3.1).

More generally we present a unified treatment of the factorization problem, and thus much of §1 is expository. There we develop the theory of a corresponding abstract factorization problem for nonnegative Hilbert space operators. Both the results and methods of §1 are purely operator theoretic. In §2 we show how the abstract theory relates to the theory of operator valued functions defined on the circle Γ or line R. The main applications to the factorization problem for nonnegative operator valued functions are deferred to §3.

The factorization problem arises in the prediction theory of stationary stochastic processes. For this connection see Helson and Lowdenslager [11], Rozanov [27], and Wiener and Masani [28].

We wish to thank Professor Loren Pitt for calling our attention to the paper by E. Robinson [23]. We have extended Robinson's results in §2.

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