BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 77, Number 2, March 1971

NONLINEAR FUNCTIONAL EQUATIONS IN BANACH SPACES AND HOMOTOPY ARGUMENTS[1]

BY PETER HESS

Communicated by Felix Browder, July 13, 1970

Let T be a (nonlinear) mapping from the real reflexive Banach space X into its conjugate space X^*. We investigate the solvability of the functional equation $Tu=0$ in a given open bounded subset Ω of X. If the mapping T is coercive, existence theorems have been derived for T belonging to various classes of nonlinear mappings of monotone type. We consider here the situation where the mapping $T=A_0$ is (in an appropriate sense) homotopic to an odd operator A_1. Taking the whole set Λ of finite-dimensional subspaces of X rather than an injective approximation scheme, we do not assume the separability of the space X. We further work with only mild continuity assumptions on the mappings involved and do not suppose their boundedness. Using an approximation of a maximal monotone mapping by single-valued, everywhere defined maximal monotone operators due to Brézis-Crandall-Pazy [3], we are able to derive results based on homotopy arguments also for multi-valued, not everywhere defined mappings. We then apply our results to asymptotically homogeneous mappings and obtain some kind of Fredholm alternative for the existence of solutions of the equation $Tu=f$. A Fredholm alternative for nonlinear equations of Hammerstein type is finally stated. Our theorems extend results by Browder [5], [6], [7], Browder-Petryshyn [10], Figueiredo-Gupta [11], Nečas [15] and others. We note that we assume for the homotopy $A(u, t): X \times [0, 1] \to X^*$ that it is continuous in t, uniformly with respect to u in bounded sets. A homotopy argument which weakens this assumption (but demands the continuity of the mappings) is given by the writer in [13].

Before stating our main results, we put together some notations and definitions we employ. If X is a real Banach space, X^* its conjugate space, (v, u) denotes the duality pairing between $v \in X^*$

AMS 1969 *subject classifications*. Primary 3930, 4780; Secondary 4610, 5540.

Key words and phrases. Nonlinear functional equation, reflexive Banach space, homotopy arguments, odd operators, unbounded mappings, maximal monotone mappings, asymptotically homogeneous operators, Fredholm alternative, Hammerstein equation.

[1] This research was partially supported by N.S.F. Grant GP-23563.