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TRANSLATION-INVARIANT LINEAR FORMS AND A FORMULA FOR THE DIRAC MEASURE

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Following Schwartz [2] we denote by $\mathfrak{D}$, $\mathcal{E}$ and $\mathcal{S}$ the complex vector spaces of all complex-valued infinitely differentiable functions ϕ on R^n where the functions of $\mathfrak{D}$ have compact supports, the functions of $\mathcal{E}$ have arbitrary supports, and the functions of $\mathcal{S}$ (along with all their derivatives) are rapidly decreasing at infinity. We equip each of these spaces with its usual locally convex topology. These spaces and their duals $\mathfrak{D}'$, $\mathcal{E}'$ and $\mathcal{S}'$ are translation-invariant in the sense that the translated function (or distribution) $\phi_h(t) \equiv \phi(t-h)$ belongs to the space whenever ϕ does. We say that a (not necessarily continuous) linear form L on any of these spaces is "translation-invariant" if $L(\phi_h) = L(\phi)$ for all ϕ in the domain space and for all h in R^n. It is, of course, well known what the *continuous* translation-invariant linear forms on these spaces are like; namely, they are either identically zero or a constant multiple of integration over R^n.

The purpose of this paper is to announce that there exists no discontinuous translation-invariant linear form on any of the six spaces $\mathfrak{D}$, $\mathcal{E}$, $\mathcal{S}$, $\mathfrak{D}'$, $\mathcal{E}'$ or $\mathcal{S}'$. That is, integration over R^n in the spaces $\mathfrak{D}$, $\mathcal{S}$ and $\mathcal{E}'$ can be characterized (up to a multiplicative constant) simply as a translation-invariant linear form. Furthermore, we obtain this result as a simple consequence of a resolution of the first derivative of the Dirac measure δ (on the real line R) into a sum of two finite differences of distributions of compact support. We state this as our main result.

THEOREM 1. *If α and β are nonzero real numbers such that α/β is irrational and not a Liouville transcendental, then there exist two (necessarily distinct) distributions S and T on R, both with compact*

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