L_p EMBEDDING AND NONLINEAR EIGENVALUE PROBLEMS FOR UNBOUNDED DOMAINS

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Let R^N denote real N-dimensional Euclidean space. Then it is a well-known fact that the imbedding of the Sobolev space $W_{1,2}(R^N)$ in $L_p(R^N)$ is bounded for $2 \leqq p \leqq 2N/(N-2)$, but is definitely not compact. Consequently the theory of critical points for general isoperimetric variational problems defined over arbitrary unbounded domains in R^N has been little investigated despite its importance. Indeed the usual proofs for the existence of even one critical point for such problems requires the verification of some compactness criterion (such as Condition C of Palais-Smale). For quadratic functionals, such as arise in the study of the spectrum of a linear elliptic partial differential operator L of order $2m$ defined on R^N, many compact imbedding theorems have been obtained in recent years [1], [2], [3]. These results yield, in turn, interesting facts concerning the discrete spectrum of L. In this note we extend these compactness results to insure the existence of critical points for certain isoperimetric variational problems arising in the study of eigenvalue problems for nonlinear elliptic partial differential equations. The existence of stationary states for nonlinear wave equations [4] provides a natural example for which our results are useful.

1. **Imbedding theorems.** Throughout this note let Ω be an arbitrary (not necessarily bounded) domain in R^N whose boundary $\partial\Omega$ is mildly smooth (say locally Lipschitzian). By $W_{s,p}(\Omega)$ we denote the Banach space of functions $u(x)$ defined on Ω such that u and all its partial derivatives up to and including order s are in $L_p(\Omega)$ (i.e. $D^\alpha u \in L_p(\Omega)$ for $|\alpha| \leqq s$). The norm in $W_{s,p}(\Omega)$ (denoted by $\| \ \|_{s,p}$) is $\|u\|_{s,p} = \{ \sum_{|\alpha| \leqq s} \|D^\alpha u\|_{L_p}^p \}^{1/p}$. In order to state results on the imbedding of $W_{s,p}(\Omega)$ we introduce the functional $M_{\alpha,p,\Omega}(w)$ for any measurable function $w(x)$ defined on Ω, $\alpha+N>0$, and $1<p<\infty$ by setting

$$M_{\alpha,p,\Omega}(w) = \sup_x \int_{|x-y|<1;\, y\in\Omega} |w(y)|^p \omega_\alpha(y)\,dy$$

where $\omega_\alpha(x) = |x|^\alpha$ for $\alpha<0$ and $x\in\Omega$; 1 for $\alpha \geqq 0$ and $x \in \Omega$.

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