THE K-SPAN OF A RIEMANN SURFACE

BY MYRON GOLDSTEIN

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In this note, we shall answer a question first posed by Sario and Oikawa in their monograph *Capacity functions* [4], and later by Rodin [3] in his Bulletin paper. The question is this. Does the class KD of harmonic functions u with finite Dirichlet integral and such that $^*du$ has vanishing periods along all dividing cycles consist only of constant functions if and only if the K-span (for $m=0$) vanishes at some point with respect to some local parameter about it? The K-span (for $m=0$) is defined to be $\partial v/\partial x|_{z=\zeta}$ where dv reproduces for the space dKD, i.e. $(du, dv)=\pi\partial u/\partial x|_{z=\zeta}$ for all $du\in dKD$, and z denotes a local variable at ζ. Note that the definition of the span depends on ζ and the choice of the local variable at ζ.

We shall answer this question in the negative by exhibiting a Riemann surface which carries nonconstant KD functions but which has the property that the K-span (for $m=0$) vanishes at each point for some choice of the local variable at that point. Note that our K-span (for $m=0$) is Rodin's 1-span.

The Riemann surface we shall construct is the same one that appears on p. 377 of my paper *Boundaries of function spaces of Riemann surfaces* [2], but, in order to aid the reader, the details of the construction will be repeated here.

Let R_0 be a hyperbolic Riemann surface which admits no nonconstant harmonic functions with finite Dirichlet integral and has a single ideal boundary component. Let $\{\gamma_n\}$ denote a sequence of analytic Jordan arcs on R_0 such that $\gamma_n\cap\gamma_m=\emptyset$ for $n\neq m$, and such that for an arbitrary compact subset K of R_0, $\gamma_n\cap K=\emptyset$ for all sufficiently large n. Let $R'=R_0-\bigcup_{n=1}^{\infty}\gamma_n$ and take the sequence $\{\gamma_n\}$ such that R' does not belong to the class $SO_{HD'}$ i.e. such that there exists a nonnegative Dirichlet function on R_0 which is harmonic on R' and vanishes quasi everywhere on $\bigcup_{n=1}^{\infty}\gamma_n$ but does not vanish quasi everywhere on R_0. Let R_1' and R_2' be two copies of R'. Denote by γ_n^+ (resp. γ_n^-) the positive (resp. negative) edge of γ_n. For each n, identify γ_n^+ of R_1' with γ_n^- of R_2' and γ_n^- of R_1' with γ_n^+ of R_2'. The resulting Riemann surface R has a single ideal boundary component. Furthermore, $R\in 0_{HD}^2-0_{HD}^1$, i.e. the dimension of the vector

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