THE NUMBER OF ROOTS OF $f(x) = a$

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Theorems 1 and 2 below are extensions of some of the results given in [2]. Some of the results used by way of introduction to these theorems and their corollaries may be found in the author's doctoral dissertation written with the helpful advice and encouragement of Robert F. Brown. Detailed proofs will be given elsewhere.

Throughout this paper $f: X \to Y$ will be a map (continuous function) from a path connected topological space X into a path connected topological space Y, and a will be a point in Y. We are interested in the number of roots $x \in X$ to the equation $f(x) = a$. Two such roots x_0 and x_1 are *equivalent* if there is a path $C: [0, 1] \to X$ from x_0 to x_1 such that $[f \circ C] = [a]$. (Here $[f \circ C]$ denotes the fixed-end-point homotopy class containing $f \circ C$ and a is used both to denote the point $a \in Y$ as well as the constant path at $a \in Y$.) This equivalence is indeed an equivalence relation; an equivalence class of roots will be called a *root class.*

Suppose $\{h_t\}$ is a homotopy of maps $h_t: X \to Y$. Then a root x_0 of $h_0(x) = a$ is said to be $\{h_t\}$ related to a root x_1 of $h_1(x) = a$ iff there is a path C in X from x_0 to x_1 such that the path D in Y defined by $D(t) = h_t(C_t)$ is fixed-end-point homotopic to a. A root x_0 of $f(x) = a$ is *essential* iff for any homotopy $\{h_t\}$ beginning at f, there is a root x_1 of $h_1(x) = a$ to which x_0 is $\{h_t\}$ related. If one root in a root class is essential, then they all are and we say that the root class itself is *essential.* The number of essential root classes is called the *Nielsen number* of (f, a) and is denoted by $N(f, a)$. (This is the Δ_2-Nielsen number of [1] and [2].) $N(f, a)$ is clearly a lower bound for the number of solutions of $f(x) = a$. If f' is homotopic to f then $N(f, a) = N(f', a)$.

The order of the cokernel of the fundamental group homomorphism $f_\#: \pi(X, x_0) \to \pi(Y, f(x_0))$ is denoted by $R(f)$; it is independent of the choice of $x_0 \in X$ and depends only on the homotopy class of f. There are always at most $R(f)$ root classes of $f(x) = a$ so, in particular, $R(f) \geqq N(f, a)$. Suppose $\{h_t\}$ is a homotopy of maps $h_t: Y \to Y$ such that h_0 is a homeomorphism leaving a fixed, and h_1 is the identity on Y. Define a loop C in Y at a by $C(t) = h_1(a)$. The group of all elements of $\pi(Y, a)$ with a representative of this form will be denoted by $S(Y, a)$. This group contains the Jiang subgroup $T(Y) \subset \pi(Y, a)$ used

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