DECOMPOSING THE BROWNIAN PATH

BY DAVID WILLIAMS

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1. Starred references are to Itô and McKean [3], the terminology of which is used here.

Because of the method of time substitution (Chapter 5*), results on the structure of the Brownian path generalise easily to provide analogous results for arbitrary diffusions. The generalisations of the results described here take simple forms in terms of scale function, speed measure and generator. These generalisations and their proofs will appear elsewhere.

Notation. A BM^0 is a (one-dimensional) Brownian motion starting at 0. By a BES^0_k we mean a 'k-dimensional' Bessel process starting at 0, i.e., a continuous random function identical in law to the radial part of a k-dimensional Brownian motion which starts at 0. See §2.7*. Introduce the k-dimensional Bessel bridge BES(k)BR as a random function on the parameter set $[0, 1]$ identical in law to

$$tr_k[(1-t)/t]$$

where r_k is a BES^0_k. Compare the 'Brownian bridge' in Billingsley [1].

2. Now let $w = \{w(t): t \geqq 0\}$ be a BM^0 and put

$$\tau = \inf\{s: w(s) = 1\}.$$

THEOREM 1. *If r_3 is a BES^0_3, then the random functions*

$$\{1 - w(\tau - t): 0 \leqq t < \tau\} \quad \text{and} \quad \{r_3(t): 0 \leqq t < \sup[s: r_3(s) = 1]\}$$

are identical in law.

Next, put

$$\sigma = \sup\{s: s < \tau; w(s) = 0\},$$
$$\alpha = \sup\{w(s): s < \sigma\}.$$

LEMMA 1. *With probability one, there is precisely one value ρ in $(0, \sigma)$ such that $w(\rho) = \alpha$.*

Theorem 2 provides a type of decomposition of the Brownian path which proves useful.

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