KOSZUL RESOLUTIONS AND THE STEENROD ALGEBRA[1]

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1. Introduction. We outline a simple procedure for constructing resolutions for a large class of algebras including the Steenrod algebra and the universal enveloping algebras. It is classical that the cohomology groups of a Lie algebra may be computed using the Koszul resolution [2, Chapter 8, §7]. This resolution is particularly nice because it is a small subcomplex of the bar resolution. Our resolutions are conceptually analogous and so we call them Koszul resolutions and the algebras for which they are defined Koszul algebras. For each such algebra A we give (Theorem 3) an explicit differential algebra whose homology is the cohomology algebra $H^*(A)$. Our theory (§3, (1), (2)) subsumes May's generalization of the classical Koszul resolution and his resolution for a restricted Lie algebra in characteristic 2 [3]. In the (motivating) case of the Steenrod algebra we also retrieve (§3, (3)) the Λ algebra of Kan et al. [1].

A detailed treatment of our results will appear in [5]. The author wishes to thank Daniel M. Kan and especially J. Peter May for several conversations and to acknowledge a debt to [1] and [3].

2. Koszul algebras and resolutions. We define Koszul algebras, give examples (2.2) and then define Koszul resolutions in (2.3). By an algebra A we shall always mean an augmented algebra, $F \xrightarrow{\eta} A \xrightarrow{\epsilon} F$, of finite type over a field F. A presentation of A is an epimorphism of algebras $\alpha: T\{x_i\} \to A$, where $T\{x_i\}$ is the tensor algebra. Let $R = \operatorname{Ker}(\alpha)$ then $A \cong T\{x_i\}/R$. Now A is said to be a *pre-Koszul* algebra if it admits a presentation α such that R is the two sided ideal generated by elements of the form $\sum_i f_i x_i + \sum_{j,k} f_{j,k} x_j x_k$ where f_i and $f_{j,k}$ are in F. Let $\alpha(x_i) = a_i$. If $\{a_i\}$ are linearly independent then $\{a_i\}$ is called a *pre-Koszul set of generators* for A and α is called a *pre-Koszul presentation*. Note that A is defined by relations of the form

$$(2.1) \qquad \sum_i f_i a_i + \sum_{j,k} f_{j,k} a_j a_k = 0.$$

A pre-Koszul algebra is said to be *homogeneous* if each $f_i = 0$ in each relation (2.1). Now $T\{x_i\}$ is filtered with $F_p T\{x_i\}$, $p \geq 0$, spanned by all

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