ARITHMETICAL PROPERTIES OF FINITE RINGS AND ALGEBRAS, AND ANALYTIC NUMBER THEORY

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The object of this note is to state certain theorems, whose proofs together with related results will appear elsewhere. The theorems are mainly concerned with asymptotic enumeration of the isomorphism classes of finite rings or finite-dimensional algebras lying in various naturally-defined categories. Two results concern enumeration of subalgebras or subrings in a given algebra or ring. All the rings and algebras are associative, but need not necessarily have units.

1. Semisimple rings and algebras.

THEOREM 1. *Let $s(n)$ denote either the total number of nonisomorphic semisimple rings of order p^n, or the total number of nonisomorphic n-dimensional semisimple algebras over the Galois field $GF(p^r)$, p a prime. Then*

$$s(n) = \exp([\tfrac{1}{3}\pi^2 + o(1)]n^{1/2}) \quad as\ n \to \infty.$$

THEOREM 2. *Let $s_{\mathfrak{R}}(n)$ denote the total number of nonisomorphic n-dimensional semisimple algebras over a real closed field $\mathfrak{R}$. Then as $n\to\infty$*

$$s_{\mathfrak{R}}(n) = \exp([b + o(1)]n^{1/3})$$

where $b = \frac{3}{4}(3+2^{1/2})^{2/3}\pi^{1/3}[\zeta(3/2)]^{2/3}$.

The proofs of these theorems, and the next one, make use of the *zeta functions* or *generating functions* of the relevant categories. In the present cases, those functions can be calculated and this is helpful in obtaining asymptotic estimates, even though usually it provides no direct asymptotic information. In particular, Theorems 1 and 2 depend on a *Tauberian theorem* of Hardy and Ramanujan [2].

The proof of Theorem 2 gives the following

COROLLARY. *Let $\pi_{\mathfrak{R}}(x)$ denote the total number of nonisomorphic*

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