ON GROUP ALGEBRAS

BY MARTHA SMITH

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For any (discrete) group G and any field F, let FG denote the group algebra of G over F. Thus elements of FG are finite formal sums $\sum a(g)g$, where $a(g) \in F$, $g \in G$.

THEOREM 1. *Suppose G is any group and F is the field of complex numbers. Let $*$ denote the involution on FG given by*

$$\left(\sum a(g)g\right)^* = \sum \overline{a(g)}g^{-1}.$$

Let Z denote the center of FG. Then there exists a function $\natural$ from FG into Z with the following properties:

(i) $(a+b)^\natural = a^\natural + b^\natural$,
(ii) $(za)^\natural = za^\natural$ *for* $z \in Z$,
(iii) $(ab)^\natural = (ba)^\natural$,
(iv) $z^\natural = z$ *for* $z \in Z$,
(v) $(a^*)^\natural = (a^\natural)^*$,
(vi) $(aa^*)^\natural = 0$ *implies* $a = 0$.

In fact, if $W(G)$ is the W^*-algebra generated by the left action of G on $l^2(G)$, then we may embed FG in $W(G)$, and then the above function is just the restriction to FG of the function $\natural$ on $W(G)$ studied by Dixmier [2].

A two-sided ideal of a ring R is said to be an annihilator ideal of R if it is the left annihilator of some subset of R.

LEMMA 2. *Let F be a field of characteristic zero and G any group. If I is an annihilator ideal of FG and a, b are elements of I such that $axb - bxa \in I$ for all $x \in FG$, then there exist elements y, z in the center of FG, and not both zero, such that $ya - zb \in I$. If further $a(FG)b \not\subseteq I$, then y and z may be chosen so that $ya \notin I$.*

COROLLARY 3. *If F is algebraically closed of characteristic zero and FG is an order in a ring Q, then the center of FG is an order in the center of Q.*

Given a group G, let Δ denote the subgroup of elements which have only finitely many conjugates in G. Let Δ^+ denote the subgroup of torsion elements of Δ. By a result of Passman [5], FG is semiprime

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