DIMENSION AND MULTIPLICITY FOR GRADED ALGEBRAS[1]

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We want to reconsider a problem that goes back to Hilbert [3]. Let $R=\sum R^p$ be a commutative algebra which is graded by the nonnegative integers and finitely generated over $R^0=F$, which for simplicity is a field. Let $M=\sum M^p$ be a finitely generated graded R-module, with p again restricted to the nonnegative integers. Each component M^p is a finite-dimensional vector space over F. If R is generated over F by elements homogeneous of degree one then Hilbert proved that there is a polynomial

$$H_M(p) = e(M)p^{n-1}/(n-1)! + \cdots$$

such that $H_M(p)=\dim M^p$ for p large. With the understanding that the zero polynomial is of degree -1, we may call n the *dimension* of M. The coefficient $e(M)$ is a nonnegative integer, the *multiplicity* of M.

Unfortunately, if R is not generated by elements of degree one, it is not usually true that $\dim M^p$ is eventually given by a polynomial in p. (For example, let $M=R=F[x]$ where x is an indeterminant of degree two.) The more general case, where the generators of R are of degree greater than one, arises naturally. We need a substitute for the Hilbert polynomial and it turns out that the Poincaré series

$$P(M) = \sum (\dim M^p)t^p$$

of the module is a good substitute. In the classical situation the relation between H_M and $P(M)$ is such that H_M is of degree at most $n-1$ if and only if $(1-t)^nP(M)$ is a polynomial in t. Moreover, if H_M is of degree exactly $n-1$ then $e(M)$ is the value of $(1-t)^nP(M)$ for $t=1$. We intend to show how these facts generalize. The details of the proofs will be given elsewhere.

In [4] Serre gave a homological treatment of dimension and multiplicity for local rings. Following Serre, we wish to define the multi-

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