THE SOLUTION OF THE PROBLEM OF INTEGRATION IN FINITE TERMS

BY ROBERT H. RISCH

Communicated by M. H. Protter, October 22, 1969

Introduction. The problem of integration in finite terms asks for an algorithm for deciding whether an elementary function has an elementary indefinite integral and for finding the integral if it does. "Elementary" is used here to denote those functions built up from the rational functions using only exponentiation, logarithms, trigonometric, inverse trigonometric and algebraic operations.

This vaguely worded question has several precise, but inequivalent formulations. The writer has devised an algorithm which solves the classical problem of Liouville. A complete account is planned for a future publication. The present note is intended to indicate some of the ideas and techniques involved.

Basic notions. We will deal exclusively with differential fields $\mathfrak{D}$ of characteristic zero; a differential field being a field endowed with a unary operation $'$ which satisfies the sum and product rule for derivatives. $\mathfrak{D}$ has a differential subfield K, called the constant field of $\mathfrak{D}$. It consists of all $\alpha \in \mathfrak{D}$ such that $\alpha' = 0$.

If $\mathfrak{D}$ is a differential subfield of $\mathfrak{F}$, then $\mathfrak{F}$ (and any $f \in \mathfrak{F}$) is said to be *elementary* over $\mathfrak{D}$ iff $\mathfrak{F} = \mathfrak{D}(\theta_1, \cdots, \theta_n)$ where each θ_i satisfies at least one of the following conditions:

(1) θ_i is algebraic over $\mathfrak{D}(\theta_1, \cdots, \theta_{i-1})$,

(2) $\theta_i'/\theta_i = f'$ for some $f \in \mathfrak{D}(\theta_1, \cdots, \theta_{i-1})$ (the exponential case),

(3) $f'/f = \theta_i'$ for some $f \in \mathfrak{D}(\theta_1, \cdots, \theta_{i-1})$ (the logarithmic case).

If in (2) or (3) θ_i is transcendental over $\mathfrak{D}(\theta_1, \cdots, \theta_{i-1})$ and the constant field of $\mathfrak{D}(\theta_1, \cdots, \theta_{i-1})$ is the same as that of $\mathfrak{D}(\theta_1, \cdots, \theta_i)$, then θ_i is a *monomial* over $\mathfrak{D}(\theta_1, \cdots, \theta_{i-1})$. If each θ_i is either algebraic or a monomial over $\mathfrak{D}(\theta_1, \cdots, \theta_{i-1})$ then $\mathfrak{F}$ is *regular elementary* over $\mathfrak{D}$.

(1), (2) and (3) are all the operations needed since we get the trigonometric and inverse trigonometric operations by adjoining $\sqrt{(-1)}$ to K.

The following basic result gives us the form assumed by elementary

AMS Subject Classifications. Primary 1280, 3402; Secondary 1451.

Key Words and Phrases. Elementary function, integration in finite terms, algorithm, differential field, Jacobian variety, bounding torsion.