COMMUTATIVE RINGS WITH IDENTITY HAVE RING TOPOLOGIES

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Throughout, let R denote a commutative ring with identity. By a *proper* topology we mean a separated nondiscrete ring topology.

THEOREM. *Every infinite R has a proper topology.*

The following five propositions outline the proof. Details will appear in [**2**].

PROPOSITION 1. *Each infinite R satisfies at least one of the following conditions.*

(a) *R admits a proper ideal topology (i.e. one having a neighborhood basis at 0 consisting of ideals).*

(b) *R contains infinitely many nilpotents.*

(c) *There is an element $r \in R$ such that $R/\mathrm{Ann}_R r$ is an infinite field.*

The proof depends on the characterizations of rings with proper ideal topologies in [**1**]. To prove the theorem we now need only consider rings satisfying (b) or (c).

PROPOSITION 2. *Let I be an ideal of R having a proper R-algebra topology $\mathfrak{I}$ (R discrete). There is a unique proper topology on R such that I (with topology $\mathfrak{I}$) is an open subspace.*

PROPOSITION 3. *Let $\phi: R/\mathrm{Ann}_R r \rightarrow (r)$ be the obvious R-module isomorphism. For each proper topology on $R/\mathrm{Ann}_R r$, ϕ induces a proper R-algebra topology on (r). Hence, if $R/\mathrm{Ann}_R r$ has a proper topology, so does R.*

It is known that all infinite fields have proper topologies [**3**, Theorem 5.2, p. 159]. With this result and Proposition 3 we have

COROLLARY. *If R satisfies* (c), *R has a proper topology.*

PROPOSITION 4. *Every infinite abelian group I has a separated nondiscrete group topology such that every endomorphism of I is continuous.*

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