MEASURES WHICH ARE CONVOLUTION EXPONENTIALS[1]

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Let $M(R)$ denote the measure algebra on the additive group of the reals. R. G. Douglas recently pointed out to us the importance of the following question in the study of Wiener-Hopf integral equations: if $\mu \in M(R)$ is invertible, then under what conditions does $\mu = \exp(\nu)$ for some $\nu \in M(R)$?

The relevance of the above question in integral equations stems from the fact that if $\mu \in M(R)$ is invertible, then μ is an exponential if and only if μ has a factorization of the form $\mu = \mu_1 * \mu_2$, where μ_1 and μ_2 are invertible elements of $M[0, \infty)$ and $M(-\infty, 0]$ respectively. In fact, if $\mu = \exp(\nu)$ and $\nu_1 = \nu|_{[0,\infty)}$, $\nu_2 = \nu|_{(-\infty,0)}$, then $\mu_1 = \exp(\nu_1)$ and $\mu_2 = \exp(\nu_2)$ yields such a factorization.

Now if W_μ is the Wiener-Hopf operator on $L^p[0, \infty)$ $(p \geqq 1)$ given by

$$W_\mu f(x) = \int_0^\infty f(y)d\mu(x-y), \tag{1}$$

then it is easy to see that W_μ is invertible if $\mu = \mu_1 * \mu_2$ with μ_1 and μ_2 invertible elements of $M[0, \infty)$ and $M(-\infty, 0]$ respectively. In fact, $W_\mu = W_{\mu_2} \circ W_{\mu_1}$, $W_{\mu_1}^{-1} = W_{\mu_1^{-1}}$, and $W_{\mu_2}^{-1} = W_{\mu_2^{-1}}$ in this case (however, it may not be true that $W_\mu = W_{\mu_1} \circ W_{\mu_2}$). Thus, if μ is an exponential, W_μ is an invertible Wiener-Hopf operator. A general survey of the invertibility problem for Wiener-Hopf operators appears in [3].

If A is a commutative Banach algebra with identity, let A^{-1} and $\exp(A)$ denote the group of invertible elements of A and the subgroup consisting of the range of the exponential function. It is well known that $\exp(A)$ is the connected component of the identity in A^{-1}. The index group of A is the factor group $A^{-1}/\exp(A)$. Arens [1] and Royden [5] have shown that this is isomorphic to the first Čech cohomology group, with integral coefficients, of the maximal ideal space of A. Our problem then, is to determine the index group of

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