SEMIGROUP PRODUCT FORMULAS AND ADDITION OF UNBOUNDED OPERATORS

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1. **Introduction.** Let A and B be selfadjoint operators on a Hilbert space. If these operators are unbounded the problem of interpreting their sum as a selfadjoint operator is not trivial. Of course, if the algebraic sum $A+B$, defined on the common domain $D(A)\cap D(B)$, is selfadjoint or has a selfadjoint closure there is no difficulty. But in general $A+B$ may have infinitely many different selfadjoint extensions; indeed, $D(A+B)$ may reduce to (0); or, on the other hand, $A+B$ need admit no selfadjoint extension.

One reason for the study of such questions arises in physics: in many quantum mechanical systems the Hamiltonian operator is the formal sum of a well-defined "free" Hamiltonian A and an "interaction" Hamiltonian B. It follows from a result of Trotter [5] that if the closure $\mathrm{cl}(A+B)=C$ is selfadjoint then the Lie product formula

$$\lim_{n\to\infty} (e^{it/nA}e^{it/nB})^n = e^{itC} \tag{1}$$

is valid. (For a proof of a more general product formula, see [1].) Interestingly, however, Nelson [4] demonstrated that the limit in (1) can exist even if $A+B$ is not essentially selfadjoint, and can successfully be used to define the dynamical group, and therefore the Hamiltonian operator, for certain quantum systems.

It is therefore of interest to study the general properties of product limits like the above, and in particular their use to define a generalized addition for unbounded operators. In this paper we shall state a number of results in this area. Proofs of these and additional results will be given elsewhere.

2. **Product limits.** Let X be a Banach space. Let $F(t)$, $0\leqq t<\infty$, be a strongly continuous function whose values are linear contraction

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