COERCIVENESS OF THE NORMAL BOUNDARY PROBLEMS FOR AN ELLIPTIC OPERATOR

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Let Ω be a bounded open subset of $\boldsymbol{R}^n$, with smooth boundary Γ (the theory is easily extended to compact manifolds). Let A be a differential operator of order $2m$ ($m \geqq 1$), with coefficients in $C^\infty(\bar{\Omega})$, such that A is uniformly strongly elliptic and formally selfadjoint in $\bar{\Omega}$. We consider the $L^2(\Omega)$-realizations of A, determined by boundary conditions of the form

$$\gamma_j u - \sum_{k \in K, k<j} F_{jk}\gamma_k u = 0, \qquad j \in J; \tag{1}$$

here J and K are complementing subsets, each consisting of m elements, of the set $M = \{0, \cdots, 2m-1\}$; the F_{jk} denote (pseudo-) differential operators in Γ of orders $j-k$; and the γ_k denote the standard boundary operators: $\gamma_0 u = u|_\Gamma$, $\gamma_k u = D_n^k u|_\Gamma$, for $u \in C^\infty(\bar{\Omega})$, where $iD_n = \partial/\partial n$ is the interior normal derivative at Γ. (1) is a reduced form of the usual *normal* type of boundary conditions, generalized to include pseudo-differential operators in Γ.

Let $\tilde{A}$ be the operator in $L^2(\Omega)$ defined by

$$\begin{aligned} D(\tilde{A}) &= \{u \in L^2(\Omega) \mid Au \in L^2(\Omega), u \text{ satisfies } (1)\}, \\ \tilde{A}u &= Au \text{ on } D(\tilde{A}). \end{aligned} \tag{2}$$

(The definition is given a sense by the general concept of boundary value introduced by Lions-Magenes [7]). We shall give below a necessary and sufficient condition on the operators F_{jk} (together with A) in order that $\tilde{A}$ be m-coercive, i.e. satisfies

$$\operatorname{Re}(\tilde{A}u, u) + \lambda\|u\|_0^2 \geqq c\|u\|_m^2, \qquad \forall u \in D(\tilde{A}),^1 \tag{3}$$

for some $c > 0$, $\lambda \in \boldsymbol{R}$. The condition has two parts:

1° it is necessary that the F_{jk} with j and $k \geqq m$ are certain functions of the F_{jk} with j and $k < m$ in order that $\tilde{A}$ be even lower bounded (Theorem 1);

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[1] Here $\|u\|_s$ denotes the norm in the Sobolev space $H^s(\Omega)$, $s \in \boldsymbol{R}$.