CONTRACTIVE PROJECTIONS AND PREDICTION OPERATORS[1]

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1. **Introduction.** The purpose of this note is to present some results on characterizations of subspaces of a general class of Banach function spaces (BFS) admitting contractive projections onto them, and to include an application to nonlinear prediction (and approximation) theory.

Let L^ρ be the subspace of all measurable scalar functions f on (Ω, Σ, μ) with $\rho(f) = \rho(|f|) < \infty$, where $\rho(\cdot)$ is a function norm, i.e., a norm with the additional properties

(i) $0 \leqq f_n \uparrow \Rightarrow \rho(f_n) \uparrow$, and

(ii) $\rho(\cdot)$ verifies the triangle inequality for infinite sums. Then L^ρ is also complete, called a BFS, (cf. [6] and [4]). It will also be assumed, for convenience, that $0 \leqq f_n \uparrow f \Rightarrow \rho(f_n) \uparrow \rho(f)$, the Fatou property. $\rho(\cdot)$ is an absolutely continuous norm (a.c.n.) if for each $f \in L^\rho$, $\rho(f\chi_{A_n}) \xrightarrow{/} 0$ for any A_n in Σ, $A_n \downarrow \varnothing$. If $\mathfrak{X}$ is a B-space, $L^\rho_{\mathfrak{X}}$ is the space of $\mathfrak{X}$-valued strongly measurable functions f on Ω, with $\rho(|f|_{\mathfrak{X}}) < \infty$, where $\rho(\cdot)$ is as above. Then $L^\rho_{\mathfrak{X}}$ is also complete. Finally let $\mathfrak{M}^\rho_{\mathfrak{X}} = \overline{\mathrm{sp}}\{fx : f \in L^\rho,\ x \in \mathfrak{X}\} \subset L^\rho_{\mathfrak{X}}$. A projection is a linear idempotent operator.

The projection problem, stated at the outset, has been first treated for $L^\rho = L^1$ in [5], and a more detailed consideration of the same case, with $\mu(\Omega) < \infty$, has been given in [2]. If $L^\rho = L^p$, also with $\mu(\Omega) < \infty$, it was then considered in [1], and these results were extended for $L^\rho = L^\Phi$, the Orlicz spaces, with a.c.n. and μ σ-finite, in [10]. The general solution of the problem in the scalar case, and a less general one in the vector case, will be given below.

2. **Contractive projections.** Let $\mathcal{S} \subset L^\rho$ be a closed subspace. If $L^\rho \neq L^2$, then, as is well known, not every $\mathcal{S}$ is the range of a bounded projection. The positive solution is given by the following result for L^ρ-spaces. (An operator T is positive if $Tf \geqq 0$ for $f \geqq 0$.)

THEOREM 1. *If (Ω, Σ, μ) is a measure space, let $L^\rho(\Sigma)$ be the BFS defined above. Consider the statements:*

(a) *$\mathcal{S}$ is the range of a (positive) contractive projection in $L^\rho(\Sigma)$.*

(b) *there is an isometric isomorphism $\Psi : L^\rho(\Sigma) \mapsto L^\rho(\Sigma)$, ($\Psi =$ identity) such that*

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