A METHOD OF ASCENT FOR SOLVING BOUNDARY VALUE PROBLEMS

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Stefan Bergman [1] and Ilya Vekua [4] have given representation formulas for solutions of the partial differential equation (1). We obtain an improvement of their results for the case of two independent variables (namely equation (2) with n set equal to 2). Furthermore, we are able to extend our result to higher dimensions (the *ascent*) by a remarkably simple variation of this two dimensional formula. Our representation (2) also contains Vekua's formulas [4, p. 59], for the Helmholtz equation in $n \geqq 2$ variables.

THEOREM 1. *Let $B(r^2)$ be an entire function of r^2, and $R(\zeta, \zeta^*; z, z^*)$ be the Riemann function*[1] *of the elliptic partial differential equation,*[2]

$$\Delta_2 u + B(r^2)u = 0, \quad r = \|\mathrm{x}\|, \quad \mathrm{x} = (x_1, x_2). \tag{1}$$

Then the function defined by

$$u(\mathrm{x}) = h(\mathrm{x}) + \int_0^1 \sigma^{n-1} G(r; 1 - \sigma^2) h(\mathrm{x}\sigma^2) d\sigma, \quad \mathrm{x} = (x_1 \cdots, x_n) \tag{2}$$

where $h(\mathrm{x})$ is harmonic in a star-like region (with respect to the origin) D, and $G(r, 1-\sigma^2) \equiv -2rR_1(r\sigma^2, 0; r, r)$, is a solution of

$$\Delta_n u + B(r^2)u = 0, \tag{3}$$

for $\mathrm{x} \in D$. Furthermore, each regular solution of (3) *may be represented in the form* (2).

PROOF. Using Bergman's integral operator of the first kind [1, p. 10], which generates a complete system of solutions for equation (1), namely

$$u(\mathrm{x}) = 2 \operatorname{Re}\left\{ \int_0^{+1} E(r, t) f(z[1 - t^2]) \frac{dt}{(1 - t^2)^{1/2}} \right\}, \quad \|\mathrm{x}\| = r \tag{4}$$

one may obtain the alternate representation,

$$u(\mathrm{x}) = h(\mathrm{x}) + \sum_{l \geq 1} 2 \frac{e_l(r^2)}{B(l, \frac{1}{2})} \int_0^1 \sigma(1 - \sigma^2)^{l-1} h(\sigma^2 \mathrm{x}) d\sigma, \tag{5}$$

[1] See [2, Chapter V], [3, Chapter III], and [4].

[2] $\Delta_n \equiv \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2 + \cdots + \partial^2/\partial x_n^2$, and $z = x_1 + ix_2$, $z^* = x_1 - ix_2$, $\zeta = \xi + i\eta$, $\zeta^* = \xi - i\eta$.