ANOTHER THEOREM ON CONVEX COMBINATIONS OF UNIMODULAR FUNCTIONS

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Let R be a finite open Riemann surface; that is, R is obtained by deleting from a compact Riemann surface a finite number of disjoint closed discs, each of which has an analytic simple closed curve as boundary. Let $A(R)$ be the algebra of functions which are continuous on the closure of R and analytic on R; $A(R)$ is a Banach space under the supremum norm. An element f of $A(R)$ will be called *inner* if $|f| = 1$ on the boundary of R. The following theorem extends the author's earlier result, where R was the unit disc in the complex plane [3].

THEOREM. *The closed convex hull of the inner functions in* $A(R)$ *is the unit ball of* $A(R)$.

The proof requires two lemmas whose proofs will be given after the proof of the theorem.

LEMMA 1. *Let* $z_1, \cdots, z_N$ *be distinct points of* R *and let* h *be an analytic function on* R *bounded by* 1. *Then there is an inner function* f *in* $A(R)$ *with* $f(z_j) = h(z_j)$ *for* $j = 1, \cdots, N$.

LEMMA 2. *Let* E *be a compact subset of the boundary of* R *of zero harmonic measure and let* μ *be a positive regular Borel measure on* E. *If* g *is a continuous function on* E *of unit modulus, then there is a sequence* $\{f_n\}$ *of inner functions in* $A(R)$ *such that*

(i) f_n *converges to* g *a.e.*μ *and*

(ii) f_n *converges uniformly to one on compact subsets of* R.

PROOF OF THE THEOREM. Let Q be the closed convex hull of the inner functions in $A(R)$. By the basic separation theorem [2, V.2.10] if Q were not equal to the unit ball of $A(R)$, there would be a measure λ which strictly separated Q from some element of the unit ball of $A(R)$. By [1, Corollary 5] the set of linear functionals on $A(R)$ which attain their norm at some element of the unit ball of $A(R)$ is dense in the dual space of $A(R)$. Hence, it suffices to prove this: if λ is a measure on B, the boundary of R, with $\|\lambda\| = 1 = \int f d\lambda$, some $f \in A(R)$, $\|f\| = 1$, then $\sup \{\operatorname{Re} \int q d\lambda : q \in Q\} = 1$.

Such a measure λ has the form

$$d\lambda = \bar{f} g \, dm + \bar{f} \, d\mu$$