AN ALGEBRAIC DUALIZATION OF FUNDAMENTAL GROUPS

BY KUO-TSAI CHEN[1]

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This note presents a construction of a Hopf algebra $\pi^1(A)$ for a given augmented commutative algebra A equipped with a derivation. Such a Hopf algebra may be taken as a dualized algebraic analogy of a fundamental group.

1. The construction of $\pi^1(A)$ is motivated by dualizing the fundamental group $\pi_1(X)$ of a differentiable manifold X with a base point x_0. Let A be the R-algebra of C^∞ functions on X equipped with the derivation d, which is the usual differentiation from A into the A-module $M=\Omega A$ of C^∞ 1-forms on X. Recall that the shuffle algebra $\mathrm{Sh}(M)$ consists of the R-module of the tensor algebra $T_R(M)$ and the shuffle multiplication $\circ$. We make $\mathrm{Sh}(M)$ a Hopf R-algebra with the comultiplication ζ: $\mathrm{Sh}(M)\to\mathrm{Sh}(M)\otimes\mathrm{Sh}(M)$ given by

$$w_1 \otimes \cdots \otimes w_r \mapsto \sum_{0\leq i\leq r} (w_1 \otimes \cdots \otimes w_i) \otimes (w_{i+1} \otimes \cdots \otimes w_r)$$

$\forall w_1, \cdots, w_r \in M$. Moreover the Hopf algebra $\mathrm{Sh}(M)$ possesses an antipode (or conjugation) j.

Denote by G the monoid of piecewise smooth loops of X with the base point x_0 under the equivalence relation of reparametrization. The monoid algebra RG is a Hopf algebra whose comultiplication Δ is given by $\Delta\alpha=\alpha\otimes\alpha$, $\forall\alpha\in G$.

Given a loop α: $[0, 1]\to X$, let $\int_\alpha w_1$ be the usual integral, and define, for $r>1$, iterated path integrals

$$\int_\alpha w_1 \cdots w_r = \int_0^1 \left(\int_{\alpha|[0,t]} w_1 \cdots w_{r-1}\right) w_r(\alpha(t), \dot\alpha(t))dt.$$

Then there is a pairing $\mathrm{Sh}(M)\times RG\to R$ such that

$$\langle w_1 \otimes \cdots \otimes w_r, \alpha\rangle = \int_\alpha w_1 \cdots w_r.$$

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